OPTIMIZATION OF FILTER BANKS USING CYCLOSTATIONARY SPECTRAL ANALYSIS

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ABSTRACT

Generally, the output of a filter bank for a stationary input signal is not stationary but cyclostationary. In this paper, by using cyclostationary spectral analysis, the spectral correlation density of this output is derived. Using this result we derive a criterion to construct an optimal 2-band perfect reconstruction filter bank which minimizes the averaged mean squared reconstruction error when the high pass band signal is dropped. By adding constrains to the filter coefficients, the biorthogonal filter bank, the conjugate quadrature filter bank and the biorthogonal linear phase filter bank are respectively obtained. Some numerical results are also presented for optimal biorthogonal and PR linear phase filter banks which are compared in terms of some performance measures.

1. INTRODUCTION

Recently there are a great deal of works concerning the multirate subband filtering method [1] [2] [3] . In this method, in the analysis part a signal is first band-pass filtered and then downsampled by decimator. These signals are transmitted to the synthesis part. In the synthesis part these signals are first upsampled by interpolator and then band-pass filtered and added. If the input is a stationary stochastic signal, then the output is no longer stationary. Actually the output signal is cyclostationary with the period M where M is the rate of decimation and interpolation. To fully characterize the output, it is necessary to know the spectral correlation density (SCD) matrix of the cyclostationary output signal.

Here we first derive this density matrix by using Gladyshev's relation [4] [5] . Then we show the fact that the output of the alias free filter banks is stationary for any stationary input and see the perfect reconstruction (PR) condition from the spectral point of view. Next the result is used to derive the averaged mean squared reconstruction error when the high pass band signal is dropped in the 2-band filter bank . This is used as a criterion to optimize the low pass filter in the analysis part under the PR condition. The criterion is a generalization of the one by Vandendorpe [6] for conjugate quadrature filters (CQF) banks. Using this criterion, some numerical results are also presented for optimal biorthogonal and linear phase PR filter banks. In terms of some criteria, the obtained filter banks are compared.

2. MULTIRATE SYSTEMS

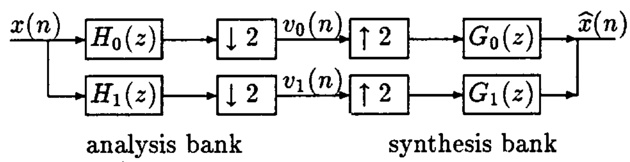

Figure 1. Filter bank ($M = 2$).

We review the multirate systems needed in this paper using the notations in [1]. Let $h_i(n)$ and $g_i(n)$ be the coefficients of the filters $H_i(z)$ and $G_i(z)$ in Fig. 1. We can write the analysis filters

$$\boldsymbol{H}(z) = \begin{pmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \end{pmatrix}^T \quad (1)$$

$$= \boldsymbol{E}_h(z^M)\begin{pmatrix} 1 & z^{-1} & \cdots & z^{-M+1} \end{pmatrix}^T \quad (2)$$

with Type 1 polyphase matrix defined by

$$(\boldsymbol{E}_h(z))_{k,l} = \sum_{n=-\infty}^{\infty} h_k(Mn+l)z^{-n} \quad (3)$$

where $(\ )_{k,l}$ is the (k,l) element of the matrix.

The alias component (AC) matrix of $\boldsymbol{H}(z)$ are defined by

$$(\boldsymbol{H}_{AC}(z))_{k,l} = H_l(zW^k), \quad W = e^{-j\frac{2\pi}{M}} \quad (4)$$

whose relation to $\boldsymbol{R}_h(z)$ is given by

$$\boldsymbol{H}_{AC}^T(z) = \boldsymbol{E}_h(z^M)\boldsymbol{\Lambda}(z)\boldsymbol{W}^\dagger \quad (5)$$

where the dagger denotes the complex conjugate transpose and $\boldsymbol{W}$ is the DFT matrix; $(\boldsymbol{W})_{k,l} = W^{kl}$ and

$$\boldsymbol{\Lambda}(z) = diag(1, z^{-1}, \ldots, z^{-M+1}). \quad (6)$$

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Define the AC matrix and the polyphase matrix of synthesis filters by the same way, denoted by $G_{AC}(z)$ and $E_g(z)$ respectively.

It is shown in [1] that the system in Fig. 1 is equivalent to the system in Fig. 2 by the polyphase matrices.

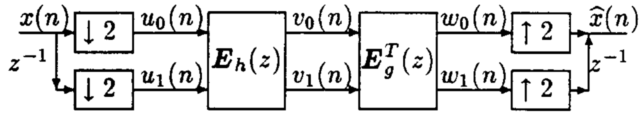

Figure 2. Filter bank using polyphase matrices.

3. CYCLOSTATIONARY PROCESSES

A process $y(n)$ with zero mean is said to be cyclostationary with period M if its covariance satisfies

$$E[y(m)y^{\dagger}(n)] = R(m,n) = R(m+M, n+M). \quad (7)$$

So for fixed u, $R(n+u,n)$ is a periodic sequence in n with period M. Thus we have the following discrete Fourier expansion

$$c_k(u) = \frac{1}{M}\sum_{n=0}^{M-1} R(n+u,n)W^{kn}. \quad (8)$$

It is stated by Gladyshev [4] that under certain conditions we can write

$$c_k(u) = \int_0^{2\pi} F_k(\omega)e^{j\omega u}d\omega. \quad (9)$$

These $F_k(\omega)$ are called the spectral correlation density (SCD) of a cyclostationary signal. It should be noted that if $y(n)$ is stationary, then $F_0(\omega)$ is the conventional spectral density and $F_k(\omega) = 0$ for $k = 1, \ldots, M-1$.

Define an M-channel process $w(n)$ by

$$w_i(n) = y(Mn+i), \quad i = 0, \ldots, M-1, \quad (10)$$

where $y(n)$ is cyclostationary with period M. It is easily shown that the multivariate process

$$\boldsymbol{w}(n) = (w_0(n), w_1(n), \ldots, w_{M-1}(n))^T \quad (11)$$

is jointly stationary. Conversely, if $\boldsymbol{w}(n)$ in (11) is stationary, $y(n)$ constructed by (10) is cyclostationary with period M. By using $\boldsymbol{W}$ and $\boldsymbol{\Lambda}(z)$ in (6), the relation between $F_k(\omega)$ $(k = 0, 1, \ldots, M-1)$ and the spectral density matrix $\boldsymbol{S}(\omega)$ of $\boldsymbol{w}(n)$ is given by

$$\boldsymbol{F}(\omega) = \frac{1}{M}\boldsymbol{W}\boldsymbol{\Lambda}(e^{j\omega})\boldsymbol{S}(M\omega)\boldsymbol{\Lambda}(e^{-j\omega})\boldsymbol{W}^{\dagger}, \quad (12)$$

where the SCD matrix $\boldsymbol{F}(\omega)$ is defined by

$$(\boldsymbol{F}(\omega))_{i,k} = F_{i-k}(\omega + i2\pi/M), \quad |\omega| \leq \pi/M. \quad (13)$$

This relation was first stated in Gladyshev [4] and was presented in this form in [5] . It should be noted that, by using this $\boldsymbol{F}(\omega)$, the averaged variance of the cyclostationary process is expressed as

$$\frac{1}{M}\sum_{n=0}^{M-1} R(n,n) = \int_0^{2\pi} F_0(\omega)d\omega = \int_{-\pi/M}^{\pi/M} \mathrm{tr}\boldsymbol{F}(\omega)d\omega. \quad (14)$$

4. OUTPUT OF FILTER BANKS

Now we derive the SCD matrix of the output $\hat{x}(n)$ of the M-band filter bank in Fig. 2 when the input is stationary with zero mean and the covariance $R_x(n)$ and the spectral density $S_x(\omega)$.

Define M-channel processes $\boldsymbol{u}(n)$, $\boldsymbol{v}(n)$ and $\boldsymbol{w}(n)$ as

$$(\boldsymbol{u}(n))_i = u_i(n) = x(Mn-i) \quad (15)$$
$$(\boldsymbol{v}(n))_i = v_i(n) \quad (16)$$
$$(\boldsymbol{w}(n))_i = w_i(n) = \hat{x}(Mn+i), \quad (17)$$

for $i = 0, \ldots, M-1$, respectively.

Since $x(n)$ is stationary, $\boldsymbol{u}(n)$ is M-channel stationary. After some computations, the relation between $S_x(\omega)$ and $\boldsymbol{S}_u(\omega)$ is given by

$$\boldsymbol{S}_u(\omega) = \frac{1}{M}\boldsymbol{\Lambda}(e^{j\frac{\omega}{M}})\boldsymbol{W}\boldsymbol{S}_x(\omega)\boldsymbol{W}^{\dagger}\boldsymbol{\Lambda}(e^{-j\frac{\omega}{M}}) \quad (18)$$

with the diagonal spectral matrix

$$(\boldsymbol{S}_x(\omega))_{i,i} = S_x((\omega + 2\pi i)/M). \quad (19)$$

Since $\boldsymbol{v}(n)$ is filtered by $\boldsymbol{E}_h(z)$, from (5) and (18), its spectral density matrix is given by

$$\boldsymbol{S}_v(\omega) = \frac{1}{M}\boldsymbol{H}^{\dagger}_{AC}(e^{-j\frac{\omega}{M}})\boldsymbol{S}_x(\omega)\boldsymbol{H}_{AC}(e^{-j\frac{\omega}{M}}). \quad (20)$$

In the synthesis filter bank, since $\boldsymbol{v}(n)$ is filtered by $\boldsymbol{E}_g^T(z)$ to produce the output $\boldsymbol{w}(n)$, its spectral density matrix is given by

$$\boldsymbol{S}_w(\omega) = \boldsymbol{E}_g^T(e^{j\omega})\boldsymbol{S}_v(\omega)\boldsymbol{E}_g(e^{-j\omega}). \quad (21)$$

Since $\boldsymbol{w}(n)$ is M-channel stationary, from the Gladyshev's relation (12), (20) and (21), $\hat{x}(n)$ is cyclostationary with period M whose SCD matrix is given by

$$\boldsymbol{F}(\omega) = \boldsymbol{T}^{\dagger}(e^{-j\omega})\boldsymbol{S}_x(M\omega)\boldsymbol{T}(e^{-j\omega}) \quad (22)$$

where

$$\boldsymbol{T}(z) = \frac{1}{M}\boldsymbol{H}_{AC}(z)\boldsymbol{G}_{AC}^T(z). \quad (23)$$

Now we show the following theorem.

Theorem

If a filter bank is alias free, then its output for any stationary input is stationary.

Proof

A filter bank is said to be alias free iff the matrix

$$\boldsymbol{P}(z) = \boldsymbol{R}_g(z)\boldsymbol{E}_h(z) \tag{24}$$

is pseudo-circulant where $\boldsymbol{R}_g(z)$ is Type 2 polyphase matrix of synthesis filters [1]. Substituting (24) into (23), after some computations, it can be shown that $\boldsymbol{T}(z)$ is diagonal. Since $\boldsymbol{S}_x(M\omega)$ is diagonal, from (22), $\boldsymbol{F}(\omega)$ is diagonal so that $\widehat{x}(n)$ is stationary □

Denote the 0th diagonal element of $\boldsymbol{T}(z)$ as $T(z)$. Then it can be shown that, from (13) and (22),

$$F_0(\omega) = |T(e^{j\omega})|^2 S_x(\omega), \quad |\omega| \leq \pi. \tag{25}$$

This shows that the alias free filter bank characterized by (24) is equivalent to a scalar linear time invariant system. Moreover, a filter bank is said to be a PR filter bank for a deterministic signal iff

$$T(z) = cz^{-n_0} \tag{26}$$

for some positive integer n_0 and some constant $c \neq 0$ [1]. From our cyclic spectral analysis, in the stochastic signal case, the output is stationary and satisfies

$$F_0(\omega) = c^2 S_x(\omega) \tag{27}$$

as it should be.

5. OPTIMIZATION OF FILTER BANKS

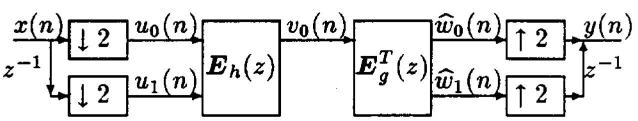

Figure 3. A system whose high pass band is dropped.

We assume that high pass bands from K to $M-1$ are dropped and denote the output of the filter bank as $y(n)$. Consider Fig. 3 ($M = 2$, $K = 1$) and define an M-channel process $\widehat{\boldsymbol{w}}(n)$ as

$$(\widehat{\boldsymbol{w}}(n))_i = \widehat{w}_i(n) = y(Mn+i) \tag{28}$$

The difference $\boldsymbol{e}(n)$ between $\boldsymbol{w}(n)$ and $\widehat{\boldsymbol{w}}(n)$ is given by

$$\boldsymbol{e}(n) = \boldsymbol{w}(n) - \widehat{\boldsymbol{w}}(n) = \boldsymbol{E}_g^T(q)I'\boldsymbol{v}(n) \tag{29}$$

where

$$I' = diag(0,\ldots,0,\overbrace{1,\ldots,1}^{K}) \tag{30}$$

and q^{-1} is the delay operator $q^{-1}x(n) = x(n-1)$.

Note that the reconstruction error

$$e(n) = \widehat{x}(n) - y(n) \tag{31}$$

satisfies $(\boldsymbol{e}(n))_i = e(Mn+i)$. From (20) and the Gladyshev's relation, the SCD matrix of $e(n)$ is given by

$$\boldsymbol{F}_e(\omega) = (\boldsymbol{T}'(e^{-j\omega}))^\dagger \boldsymbol{S}_x(M\omega)\boldsymbol{T}'(e^{-j\omega}) \tag{32}$$

for $|\omega| \leq \pi/M$ where

$$\boldsymbol{T}'(z) = \frac{1}{M}\boldsymbol{H}_{AC}(z)I'\boldsymbol{G}_{AC}^T(z). \tag{33}$$

When $M = 2$ and the high pass band signal is dropped, from (14) and (32), the averaged variance of the reconstruction error reduces to

$$\sigma_e^2 = \frac{1}{4}\int_0^{2\pi} \left(|G_1(e^{j\omega})|^2 + |G_1(e^{j(\omega+\pi)})|^2\right) \cdot |H_1(e^{j\omega})|^2 S_x(\omega)d\omega. \tag{34}$$

Therefore by minimizing the above equation under the PR condition, optimal PR filter banks can be obtained.

Now let us consider the PR filter bank. Let $H_0(z)$ and $H_1(z)$ be causal FIR with odd order N and put

$$G_0(z) = H_1(-z), \qquad G_1(z) = -H_0(-z). \tag{35}$$

Then, the PR condition (26) reduces to

$$T(z) = \frac{1}{2}(H_1(z)G_1(z) - H_1(-z)G_1(-z)) = cz^{-n_0}. \tag{36}$$

This filter bank is also called biorthogonal [3].

The filter bank which satisfies (35), (36) and

$$h_1(n) = (-1)h_0(N-n), n = 0,\ldots,N \tag{37}$$

are said to be CQF bank, PR-QMF or orthogonal. In this case, $n_0 = N$ and the criterion (34) reduces to

$$\sigma_e^2 = \frac{1}{2}\int_0^{\pi} |H_1(e^{j\omega})|^2 S_x(\omega)d\omega, \tag{38}$$

which is originally considered in [6].

In some application it is desirable to use a PR filter bank whose filters have linear phase, abbreviated PR-LPF in this paper. It is shown in [3] that in addition to (35) and (36) the coefficients must satisfy

$$h_0(n) = h_0(N-n),\ h_1(n) = -h_1(N-n), n = 0,\ldots,N. \tag{39}$$

Using the polyphase representation, from (39), we have

$$H_i(z) = E_{i0}(z^2) + (-1)^i E_{i0}(z^{-2})z^{-N}, i = 0,1 \tag{40}$$

In this case, $n_0 = N$, the PR condition and the criterion (34) reduce to

$$E_{00}(z)E_{10}(z^{-1}) + E_{00}(z^{-1})E_{10}(z) = 1, \tag{41}$$

$$\sigma_e^2 = \int_0^{2\pi} |E_{00}(e^{j2\omega})H_1(e^{j\omega})|^2 S_x(\omega)d\omega \tag{42}$$

respectively.

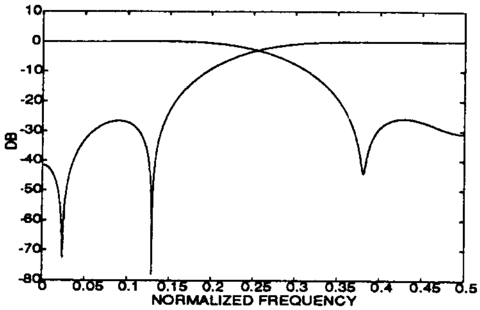

Figure 4. Frequency responses $H_0(e^{j\omega})$ and $H_1(e^{j\omega})$ of an 8-tap optimal biorthogonal bank.

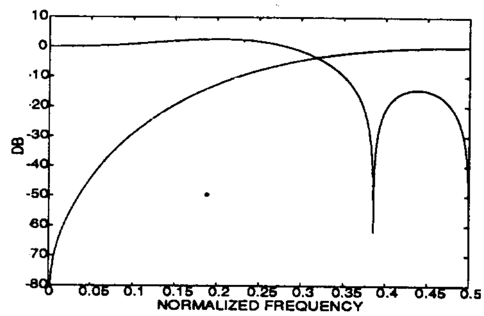

Figure 5. Frequency responses $H_0(e^{j\omega})$ and $H_1(e^{j\omega})$ an 8-tap optimal PR linear phase filters bank.

$N=7$	CQF	PR-LPF	Biorthogonal
$\eta(\%)$	96.497	96.419	96.501
β (l=2)	0.0389	0.0173	0.0373
G_{SBC} (dB)	4.34	4.82	4.41

Table 1. Properties of the optimal PR filter banks

6. RESULTS

For $R_x(n) = 0.9^{-|n|}$ and $N = 7$, the frequency responses $H_0(z)$ and $H_1(z)$ of the obtained optimal biorthogonal bank ($n_0 = N$) and PR-LPF are shown in Fig. 4 and Fig. 5, respectively. We use the coefficients of the optimal CQF bank as the initial values for nonlinear optimization, so it may be suboptimal.

A measure of our criterion is defined by

$$\eta = 1 - \sigma_e^2/\sigma_x^2 \tag{43}$$

which shows the efficiency of the compression of the input signal.

In using the PR filter bank, it is often the case that the subband signals have small correlations with each other. A measure of this is given by

$$\beta = (\sum_{\tau} \sum_{i,k=0,i\neq k}^{M-1} |(\boldsymbol{R}_v(\tau))_{i,k}|^2)^{\frac{1}{2}}/\sigma_x^2 \tag{44}$$

where $\boldsymbol{R}_v(\tau)$ is the covariance of $v(n)$. The 2-band CQF filter bank which minimizes the above measure is considered in [7].

Usually, between analysis bank and synthesis bank, there are quantizing operations. The effect of the reconstruction error due to the quantization is measured by the coding gain given in [8]

$$G_{SBC} = \frac{\sigma_x^2}{(\Pi_{i=0}^{M-1}\sigma_{v_i}^2 \sum_{n=0}^{N}|g_i(n)|^2)^{\frac{1}{M}}} \tag{45}$$

Table 1 shows the above measures. In terms of η, the biorthogonal filter bank is best as it should be and PR-LPF is worst. It is interesting that in terms of β and G_{SBC}, PR-LPF is best. In the CQF, it can be shown that the optimal bank in terms of η is equal to the optimal bank in terms of G_{SBC}.

7. CONCLUSION

In this paper, the spectral correlation density of the output of filter banks is derived. A criterion is also derived to construct optimal 2-band perfect reconstruction filter banks which minimize the averaged mean squared reconstruction error when the high pass band signal is dropped.

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