ITERATIVE ALGORITHMS FOR ENVELOPE CONSTRAINED FILTER DESIGN

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ABSTRACT

The discrete-time envelope constrained (EC) filtering problem can be formulated as a quadratic programming (QP) problem with linear inequality constraints. In this paper, the QP problem is approximated by an unconstrained minimization problem with two parameters. These parameters can be selected so that given an acceptable deviation from the norm of the optimal EC filter, the solution to the unconstrained problem satisfies both the deviation and envelope constraints. Newton's method with line search is applied to solve the unconstrained problem iteratively.

1. INTRODUCTION

Envelope constrained (EC) filter design is concerned with determining the finite impulse response (FIR) of a filter so that its noiseless response to a specified input signal lies within a given envelope (see figure 1), while minimizing the effect of input noise [1]. In a variety of signal processing fields such as communication channel equalization [1], radar and sonar detection [2], robust antenna and filter design [3], seismology [4], EC filters are more directly relevant than least square filters.

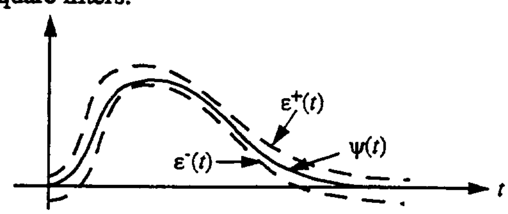

Figure 1. Output mask and noiseless output.

In [5], the EC filtering problem has been formulated as a quadratic programing (QP) problem with linear inequality constraints. Sequential quadratic programming via active set strategy is a popular tool for solving QP problems. However, implementation-wise, this is not suitable for adaptive filters. In [6], the primal-dual algorithm is used to solve the EC filtering problem adaptively. Numerical results have shown that the algorithm converges, though at a rather slow rate. In [7] contraction mappings and appropriate projections are used to develope constrained adaptive filtering algorithms, however this is only applicable where the filters are constrained to a bounded hypercube.

In this note, a new class of iterative algorithms for solving the EC filtering problem is introduced. These algorithms have better convergence characteristics which can be predicted analytically. The EC filtering problem is approximated by an unconstrained optimization problem with two undetermined parameters. The Newton-Ralphson (NR) method can then be applied to solve the unconstrained problem.

It is shown that the error between the solution to the approximate problem and the optimal filter can be made vanishingly small. A quadratic rate of convergence for a certain type of algorithm is established when the NR method is applied to solve the unconstrained problem. These results are not restricted to the EC filtering problem but also apply to other disciplines such as robust pole placement [8] which can be posed as a QP problem with linear inequality constraints.

This is the first step towards developing adaptive algorithms for EC filters. The study of the behavior of the new iterative algorithms with noisy measurements is under way.

2. THE EC FILTERING PROBLEM

Let $u = [u_1, \ldots, u_n]^T \in R^n$ be the finite impulse response of a time invariant FIR filter. Then the output of the filter due to a finite support input signal $s = [s_1, \ldots, s_m]^T \in R^m$ is $\psi = Su \in R^N$, where $N = m + n - 1$ and S is the $N \times n$ convolution matrix, given by

$$S = \begin{bmatrix} s_1 & 0 & \ldots & 0 \\ \vdots & s_1 & & \vdots \\ s_m & \vdots & \ddots & 0 \\ 0 & s_m & & s_1 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & s_m \end{bmatrix}$$

It is required that the noiseless output ψ lies within the upper and lower boundaries $\varepsilon^+, \varepsilon^- \in R^N$ of output mask, i.e.

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$\varepsilon^- \le Su \le \varepsilon^+$ (see figure 1). Let $A = [S^T, -S^T]^T \in R^{2N\times n}$ and $b = [\varepsilon^{+T}, -\varepsilon^{-T}]^T \in R^{2N}$. Then the envelope constraints, becomes $Au \le b$. The envelope constrained filter problem is posed as problem (P), given as follows

$$\min \|u\|^2 \text{ subject to } Au - b \le 0 .$$

This problem, has a unique solution, u^*, if the feasible set, $\mathcal{U} \equiv \{u \in R^n : \varepsilon^- \le Su \le \varepsilon^+\}$, is not empty. The topological properties of $\mathcal{U}$ are given in the following Lemma.

Lemma 2.1. *The set $\mathcal{U}$ of feasible filters is convex and compact. Moreover $\mathcal{U}$ is contained in a closed ball centred at the origin and radius $\|r\|/\sigma(S)$, where $r \in R^N$ is a vector with components defined by $r_j = \max\{|\varepsilon_j^-|, |\varepsilon_j^+|\}$ and $\sigma(S)$ denotes the minimum singular value of the convolution matrix S. (See [9] for proof).*

To eliminate the trivial solution, it is assumed that there exists at least one $j \in \{1, ..., N\}$ such that the product $\varepsilon_j^+ \varepsilon_j^- > 0$, where $\varepsilon_j^+, \varepsilon_j^-$ are, respectively, the j^{th} component of the upper and lower boundaries.

3. THE APPROXIMATE PROBLEM.

Suppose g_υ is a continuous function such that $g_\upsilon(x) = 0$ for all $x \le -\upsilon$ and $g_\upsilon(x)$ is strictly increasing for all $x > -\upsilon$. Let $\phi_j(u) \equiv a_j^T u - b_j$, where a_j^T denote the j^{th} row of A and let $G_{j,\upsilon}(u) = g_\upsilon(\phi_j(u))$. Then for each $\upsilon, \gamma > 0$, an augmented cost function is defined as follows

$$f_{\upsilon,\gamma}(u) \equiv \|u\|^2 + \gamma \sum_{j=1}^{2N} G_{j,\upsilon}(u)$$

The function g_υ, called a penalty allocator, determines the analytical properties of the augmented cost $f_{\upsilon,\gamma}$ as demonstrated by the following theorem (see [9] for proof).

Theorem 3.1.

(i) $f_{\upsilon,\gamma}$, is continuous and strictly convex.

(ii) If g_υ is once continuously differentiable, then $f_{\upsilon,\gamma}$ is once continuously differentiable and its gradient is

$$\nabla f_{\upsilon,\gamma}(u) = 2u + \gamma \sum_{j=1}^{2N} g'_\upsilon(\phi_j(u)) a_j .$$

(iii) If g_υ is twice continuously differentiable, then $f_{\upsilon,\gamma}$ is twice continuously differentiable and its Hessian is

$$\nabla^2 f_{\upsilon,\gamma}(u) = 2I + \gamma \sum_{j=1}^{2N} g''_\upsilon(\phi_j(u)) a_j a_j^T .$$

Hence the unconstrained minimization problem $(P_{\upsilon,\gamma})$ with $\upsilon, \gamma > 0$, defined by,

$$\min f_{\upsilon,\gamma}(u)$$

has a unique solution, which is denoted by $u^*_{\upsilon,\gamma}$. The use of $(P_{\upsilon,\gamma})$ to approximate (P) is justified by the following theorem (see [9] for proof).

Theorem 3.2. *If $0 < \upsilon \le \hat{\upsilon}$ and $\gamma > \|u_\upsilon\|^2 / g_\upsilon(0)$ for some $u_\upsilon \in \mathcal{U}_\upsilon \equiv \{u : G_{j,\upsilon}(u) = 0, j = 1, ..., 2N\}$. Then $u^*_{\upsilon,\gamma} \in \mathcal{U}$, moreover, for each $0 < \varepsilon \le 2\|\bar{u}\|^2$ where $b - A\bar{u} > 0$, we have*

$$0 < \upsilon \le \frac{\varepsilon \min_{j=1,...,2N}(b - A\bar{u})_j}{2\|\bar{u}\|^2} \Rightarrow 0 \le \|u^*_{\upsilon,\gamma}\|^2 - \|u^*\|^2 \le \varepsilon .$$

Thus for values of υ such that $\mathcal{U}_\upsilon$ is non-empty, by choosing a value of γ such that $\gamma > \|u_\upsilon\|^2 / g_\upsilon(0)$ for some $u_\upsilon \in \mathcal{U}_\upsilon$, the above theorem asserts that the approximate optimal filter $u^*_{\upsilon,\gamma}$, satisfies the envelope constraints. Moreover, for any given ε, the parameter υ can be chosen (without using any information on $u^*_{\upsilon,\gamma}$ and u^*) so that the impulse response power deviation (IRPD) defined as $\|u^*_{\upsilon,\gamma}\|^2 - \|u^*\|^2$, is less than ε. From this, it can be easily deduced that the error $\|u^*_{\upsilon,\gamma} - u^*\|$ vanishes as υ tends to zero. These results can be easily extended to a more general quadratic cost with only minor modifications.

4. ITERATIVE ALGORITHMS

Having selected υ and γ, the NR method can be applied to determine the solution iteratively. The filter coefficients u_k at the k^{th} iteration can be updated by virtue of Theorem 3.1 and the following iterative equations.

$$u_{k+1} = u_k - t_k H_k^{-1} \nabla f_{\upsilon,\gamma}(u_k) , H_k = \nabla^2 f_{\upsilon,\gamma}(u_k)$$

Theorem 4.1. *Suppose that g_υ is twice continuously differentiable for each $\upsilon > 0$ with $|g''_\upsilon(x)| \le h(\upsilon)$, $\forall x \in R$. Then, any initial estimate u_0 will converge to the solution $u^*_{\upsilon,\gamma}$ of problem $(P_{\upsilon,\gamma})$ under the NR algorithm with constant step size $t_k = t$ if*

$$0 < t < \frac{2}{1 + n\gamma h(\upsilon)\|s\|^2} .$$

The proof can be found in [9].

Considerably faster convergence can be achieved by using line search to determine a suitable step size. An effective form of line search involves the Goldstein condition [10], [11]:

$$f_{\upsilon,\gamma}(u_{k+1}) - f_{\upsilon,\gamma}(u_k) < \alpha t_k \nabla f_{\upsilon,\gamma}(u_k)^T d_k, \alpha \in (0, 0.5) .$$

and the Wolfe-Powell condition [12]

$$\nabla f_{\upsilon,\gamma}(u_k + t d_k)^T d_k \ge \beta \nabla f_{\upsilon,\gamma}(u_k)^T d_k, \beta \in (\alpha, 1)$$

One such implementation is coded as follows

```
line_search
t_upp = INF; t_low = 0; t = 1; ADMISSIBLE = FALSE;
while ADMISSIBLE == FALSE
   if GOLDSTEIN == TRUE,
      if WOLFE_POWELL == TRUE,
         ADMISSIBLE = TRUE;
      else
         t_low = t;
```

```
        if t_upp = Inf, t = Increase(t);
        else t = Refine(t, t_low, t_upp);
        end
     end
  else
     t_upp = t;
     if t_low == 0, t = Decrease(t);
     else t = Refine(t, t_low, t_upp);
     end
  end
end
```

Theorem 4.2. Suppose that g_υ is twice continuously differentiable for each $\upsilon > 0$ with $|g''_\upsilon(x)| \le h(\upsilon)$. If $g''_\upsilon(x)$ is Lipschitz continuous, i.e. $|g''_\upsilon(x) - g''_\upsilon(y)| \le L|x-y|$, $\forall x, y \in R$, and the step size t_k is chosen by the above line search procedure, then, any initial estimate u_0 will converge to the solution $u^*_{\upsilon,\gamma}$ of problem $(P_{\upsilon,\gamma})$ at a quadratic rate, under the NR algorithm. (see [9] for proof)

5. A NUMERICAL EXAMPLE

As an example, consider the compression of a rectangular pulse which has applications, for instance, in a high resolution radar system. For a rectangular input pulse represented by $s = [1, \ldots, 1]^T \in R^{10}$, using a filter with $n = 21$ coefficients and an output mask with an allowable sidelobe level of ± 0.25 and mainlobe peak of 1 ± 0.15, i.e.

$$\varepsilon^+ = [\underbrace{0.25, \ldots, 0.25}_{14}, 1.15, \underbrace{0.25, \ldots, 0.25}_{15}]^T \in R^{30}$$

$$\varepsilon^- = [\underbrace{-0.25, \ldots, -0.25}_{14}, 0.85, \underbrace{-0.25, \ldots, -0.25}_{15}]^T \in R^{30}$$

The coefficients of the 'exact' optimal EC filter u^* (obtained by QP from the MATLAB optimization toolbox) and the response of the optimal filter due to the rectangular pulse are shown in Figure 2.

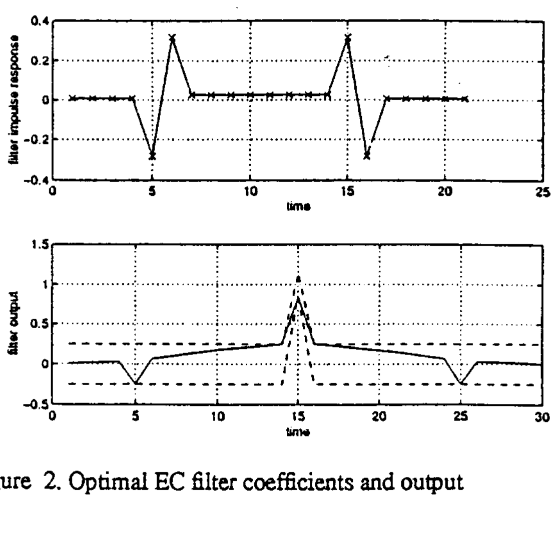

Figure 2. Optimal EC filter coefficients and output

Also examined in this example is how well the unconstrained problems approximate the EC filtering problem. To do this, it is assumed that the numerical errors associated with the exact solutions of each problems are negligible in comparison to the approximation errors.

There is a $\bar{u}$ such that $\min_{j=1,\ldots,2N}(b - A\bar{u})_j = 0.0667$, and $\|\bar{u}\|^2 = 0.6386$. Hence, in order to have an impulse response power deviation (IRPD) of less than 4×10^{-4} from the optimal EC filter, it follows from Theorem 3.2 that the accuracy parameter υ is required to be less than 2.088×10^{-5}. For convenience, let $\upsilon = 2 \times 10^{-5}$, the penalty parameters are selected by using Theorem 3.2 and the bound on u suggested in Lemma 2.1. The NR method with the given line search is used to solve the resulting unconstrained problem. The iteration process starts with the origin as an initial guess and stops when the gradient is less than 10^{-14}. Three penalty allocators are used for comparison in this example.

The first of these penalty allocators, given by (1), generates the approximate problem used in [13] which was solved by the SD and NR methods with constant step size.

$$g_\upsilon(x) = \begin{cases} 0, & x \le -\upsilon \\ (x+\upsilon)^2/4\upsilon, & -\upsilon \le x \le \upsilon \\ x, & x \ge \upsilon \end{cases} \qquad (1)$$

Note that (1) has two discontinuities in the second derivative leading to difficulties in convergence rate analysis. The IRP is plotted against the number of function evaluations or iterations in Figure 3.

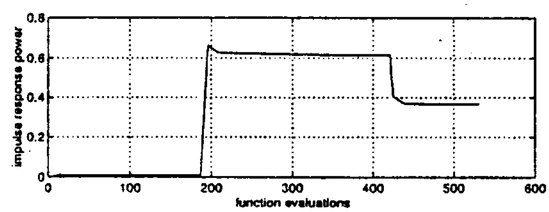

Figure 3. Convergence characteristic of penalty allocator (1)

The second penalty allocator is given by (2).

$$g_\upsilon(x) = \begin{cases} 0, & x \le -\upsilon \\ (x+\upsilon)^2/4\upsilon, & -\upsilon \le x \end{cases} \qquad (2)$$

This penalty allocator has a discontinuity in the second derivative and thus suffers from the same analytical drawback as penalty allocator (1), but displays, in practice, a phenomenal improvement in performance (see figure 4).

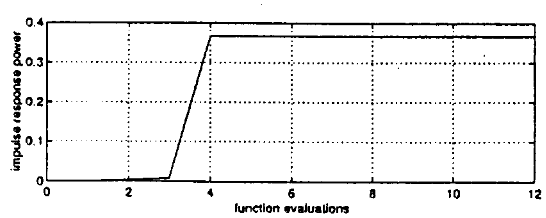

Figure 4. Convergence characteristic of penalty allocator (2)

The third penalty allocator, given by (3), is constructed to emulate penalty allocator (2) while eliminating the discontinuity in the second derivative.

$$g_\upsilon(x) = \begin{cases} 0, & x \le -\upsilon \\ -\frac{\upsilon}{\pi}\cos\left(\frac{\pi x}{2\upsilon}\right) + \frac{(x+\upsilon)}{2}, & -\upsilon \le x \le 0 \\ \frac{\pi x^2}{8\upsilon} + \frac{x}{2} + \upsilon\left(\frac{1}{2} - \frac{1}{\pi}\right), & x \ge 0 \end{cases} \quad (3)$$

From Theorem 4.2, a quadratic rate of convergence is expected when the NR method with the given line search is applied to determine the solution of the resulting unconstrained problem. The behavior of (3) is shown in Figure 5.

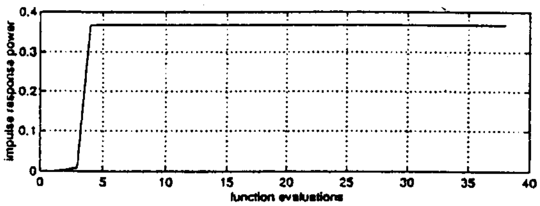

Figure 5. Convergence characteristic of penalty allocator (3)

The simulation results are tabulated in the following table

Table 1: Penalty parameters

$\upsilon = 2\text{x}10^{-5}$	penalty allocator (1)	penalty allocator (2)	penalty allocator (3)
γ	$2.4736\text{x}10^{6}$	$2.4736\text{x}10^{6}$	$3.4037\text{x}10^{6}$
IRPD	$4.8104\text{x}10^{-5}$	$4.8104\text{x}10^{-5}$	$4.8072\text{x}10^{-5}$
IRPD/$\|u^*\|^2$	$1.3090\text{x}10^{-4}$	$1.3090\text{x}10^{-4}$	$1.3082\text{x}10^{-4}$
no. f. evaluations	482	6	19
no. SD. evaluations	65	5	18

From Table 1 and Figures 3, 4 and 5 which plot the augmented cost function and the impulse response power of the filter against the number of function evaluations, the convergence behavior for each of the penalty allocators can be compared. For penalty allocators (2) and (3), only a few iterations are required, while penalty allocator (1) takes considerably many more iterations and line search calculations. In all, this is an improvement over the primal-dual algorithm which takes several thousand iterations (see [13]). Table 1 also shows that the IRPD of the filters yielded by each of the penalty allocators are about 12% of the prescribed tolerance of 4×10^{-4}. This suggests that the IRPD bound of Theorem 3.2 is very conservative.

6. CONCLUSIONS

The error estimate ε, although conservative, is still very useful as guide lines for determining the parameters υ and γ of the approximate problem. Considerably smaller values for the penalty parameter γ may be used as the bound on the feasible set suggested by Lemma 2.1 is quite loose. The simulation results demonstrated that the new algorithm based on penalty allocator (3) exhibit the convergence characteristics predicted by Theorem 4.2. The unexpected speedy convergence for penalty allocator (2) does not render it superior to (3) as there are examples where it is slower. In general, penalty allocator (3) performs better than the other two.

REFERENCES

[1] R. Lucky, J. Salz, and E. J. Weldon, *Principles of data communications*, New York: McGraw-Hill, 1968.

[2] R. J. McAulay and J. R. Johnson, "Optimal mismatched filter design for radar ranging detection and resolution", *IEEE Trans. Information Theory*, Vol. IT-17, pp. 696-701, 1971.

[3] K. M. Ahmed and R. J. Evans, "An adaptive array processor with robustness and broadband capabilities", *IEEE Trans. Antennas and Propagat.*, Vol. AP-32, pp. 944-950, 1984.

[4] L. C. Wood and S. Treitel, "Seismic signal processing", *Proc. IEEE* (Special issue on digital signal processing) Vol 63, pp. 649-661, 1975.

[5] R. J. Evans, T. E. Fortmann, and A. Cantoni, "Envelope-constrained filter, Part I, theory and applications", *IEEE Trans. Information Theory*, Vol. IT-23, pp. 421-434, 1977.

[6] R. J. Evans, A. Cantoni, and T. E. Fortmann, "Envelope-constrained filter, Part II, adaptive structures", *IEEE Trans. Information Theory*, Vol. IT-23, pp. 435-444, 1977.

[7] A. Krieger and E. Masry, "Constrained adaptive filtering algorithms: asymptotic convergence properties for dependent data", *IEEE Trans. Information Theory*, Vol. IT-35, pp. 1166-1176, 1989.

[8] W. X. Zheng, K. L. Teo, and A. Cantoni, "Robust pole assignment: revisited", *12th World Congress IFAC*, Vol 1, pp. 471-474, 1993.

[9] B. Vo, A. Cantoni and K. L. Teo, "A penalty approach to iterative algorithms for envelope constrained filter design", *Technical Report* SPL-TM 004, ATRI, Curtin University of Technology, Western Australia (1994).

[10] R. Fletcher, *Practical methods of optimizations*, 2nd edition, John Wiley & Sons, 1987.

[11] B. N. Pshenichni and Y. M. Danilin, *Numerical methods in Extremal Problems*, Mir Publishers, Moscow, 1978.

[12] DENNIS Jr., J. E., and SCHNABEL, R. B., "A view of unconstrained optimization", *Handbooks in operations research and management science*, G. L. Nemhauser et al., Eds., Elsevier Science Publishers B. V. (North-Holland) Vol. 1, pp. 1-72, 1989.

[13] W. X. Zheng, A. Cantoni, B. Vo and K. L. Teo, "Recursive procedures for constrained optimization problems and its application in signal processing", *Technical Report*, ATRI, Curtin University of Technology, Western Australia (1994).