LAGUERRE DIGITAL FILTER DESIGN

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ABSTRACT

Using Laguerre networks in discrete-time (DT) domain has already been addressed by the author [1]. The main contribution of this paper is to use the *Laguerre* networks in frequency domain, and to design the digital filters based on the specified frequency response. This filter design yields appropriate *linear phase*, with *lower* ripples in stopband and passband compared to the conventional FIR filters. Both, *analytical* and *numerical* approaches for Laguerre digital filter design will be introduced in details and the results will be shown through some examples. The procedure is based on minimizing the *mean-square-error (MSE)* between the frequency response of the desired filter and its corresponding Laguerre network frequency response.

1. INTRODUCTION

Discrete-time Laguerre functions make a complete orthogonal set with the following Z-transform [2]

$$L_k(z,b)=\sqrt{1-b^2}\,\frac{(z^{-1}-b)^k}{(1-b\,z^{-1})^{k+1}} \qquad |b|<1 \tag{1}$$

where b is an *adjustable* parameter. Thus, any linear discrete-time invariant system with specified frequency response can be represented as

$$\tilde{H}(z)=\frac{Y(z)}{X(z)}=\sum_{k=0}^{\infty} c_k(b)\,L_k(z,b) \tag{2}$$

in which $X(z)$ and $Y(z)$ are input and output Z-transforms of system respectively, and $c_k(b)$'s are known as *Laguerre coefficients.* From (1) and (2), the system Laguerre equivalent network is realized, and it is shown in Fig. 1. Consequently, we obtain

$$Y(z)=\sum_{k=0}^{\infty} c_k\,V_k(z) \tag{3}$$

where

$$\begin{aligned} V_k(z) &= L_k(z,b)\,X(z) \qquad k=0,1,\ldots \\ &= \sqrt{1-b^2}\,\frac{(z^{-1}-b)^k}{(1-b\,z^{-1})^{k+1}}\,X(z) \end{aligned} \tag{4}$$

from (4) and Fig. 1., it is obvious that the first stage is a *simple low-pass* filter and the remaining stages are *identical all-pass* filters. In physical applications, the frequency response of the system can be approximated by $L+1$ Laguerre stages as

$$H(z)=\sum_{k=0}^{L} c_k(b)\,L_k(z,b) \tag{5}$$

where we obtain the *optimum* value of b by minimizing MSE between $H(z)$ and $\tilde{H}(z)$ in frequency domain.

2. ANALYTICAL METHOD

The orthogonality property of a real set of basis functions such as $\{L_k(e^{j\omega},b)\}$ in frequency domain is

$$\frac{1}{2\pi}\int_{2\pi} L_i(e^{j\omega},b)\,L_k(e^{j\omega},b)\,d\omega=\delta_{ik} \tag{6}$$

Using this property, it can be shown that the MSE in frequency domain is

$$\begin{aligned} \varepsilon(b) &= \frac{1}{2\pi}\int_{2\pi} |\tilde{H}(\omega)-H(\omega)|^2\,d\omega \\ &= \frac{1}{2\pi}\int_{2\pi} |\tilde{H}(\omega)|^2\,d\omega-\sum_{k=0}^{L}|c_k(b)|^2 \end{aligned} \tag{7}$$

To minimize $\varepsilon(b)$ with respect to b, we set its derivative with respect to b to zero, to obtain

$$\sum_{k=0}^{L} c_k(b)\,\frac{d}{db}c_k(b)=0 \tag{8}$$

In general for any $H(z)$ or $H(\omega)$, it can be shown that (8) is equivalent to (see Appendix A)

$$\sum_{k=0}^{L} c_k(b)\,\frac{d}{db}c_k(b)=\frac{L+1}{1-b^2}c_L(b)\,c_{L+1}(b)=0 \qquad |b|<1 \tag{9}$$

Thus, b_{opt} is one of the roots of $c_L(b)=0$ or $c_{L+1}(b)=0$. Because of *multimodality* nature of $\varepsilon(b)$, we have several roots in the interval [-1,1]. For extracting these roots,

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the most reliable method is *bisection* method. b_{opt} is the one that gives $\varepsilon_{\min}(b)$. Using the orthogonality property of Laguerre functions and (1), it is shown that [1]

$$\begin{aligned} c_k(b) &= \frac{1}{2\pi j}\oint_C \tilde{H}(z)\, L_k(z^{-1},b)\, z^{-1}\, dz \qquad (10) \\ &= \frac{\sqrt{1-b^2}}{2\pi j}\oint_C \tilde{H}(z)\frac{(z-b)^k}{(1-bz)^{k+1}}\, z^{-1}\, dz \end{aligned}$$

where C is the unit circle. By inserting $\tilde{H}(z)$ in (10), we can obtain c_k's for any specified filter.

For instance, the desired frequency response for linear phase low-pass filter (LPF) with *time delay* τ, by using the change of variable $z=e^{j\omega}$ is as

$$\tilde{H}_{LP}(z) = \begin{cases} z^{-\tau} & e^{-j\omega_c} < z < e^{j\omega_c} \\ 0 & \text{otherwise on unit circle} \end{cases} \qquad (11)$$

By introducing the appropriate frequency responses for high-pass (HP), band-pass (BP) and band-reject (BR) filters and substituting the following binomial expansions

$$(1-b\,z)^{-(k+1)} = \sum_{i=0}^{\infty}\binom{k+i}{i} b^i z^i \qquad |z|<\frac{1}{b} \quad (12)$$

$$(z-b)^k = \sum_{m=0}^{k}\binom{k}{m} z^{k-m}(-b)^m \qquad (13)$$

in integrand of equation (10), we obtain

$$\begin{aligned} c_k(b) &= \sqrt{1-b^2}\sum_{p=0}^{k}\sum_{n=p}^{2\tau} s(n)\,(-1)^{k+p}\binom{k}{k-p} \qquad (14) \\ &\binom{k+n-p}{n-p} b^{k+n-2p} = \sqrt{1-b^2}\, P_k(b) \end{aligned}$$

where τ is the time delay and $P_k(b)$ is a polynomial of degree $k+2\tau$, and

$$s(n) = \begin{cases} (-1)^m \frac{\sin[\omega_2(n-\tau)]-\sin[\omega_1(n-\tau)]}{\pi(n-\tau)} & n \neq \tau \\ A + (-1)^m \frac{\omega_2-\omega_1}{\pi} & n=\tau \end{cases} \qquad (15)$$

$m=A=0$ for LPF, HPF, BPF, and $m=A=1$ for BRF. ω_1 is lower edge cutoff frequency (zero for LPF) and ω_2 is upper edge cutoff frequency (π for HPF). Now, we illustrate this method by an example.

Example 1.

We want to design a digital Laguerre filter which realizes an ideal linear phase LPF with $\omega_c=\pi/4$ and $\tau=16$. Using a Laguerre network with 20 stages ($L=19$), we use (14) and (15), and clearly find the corresponding $P_k(b)$'s for $c_{19}(b)=0$ and $c_{20}(b)=0$ that are polynomials of degree 51 and 52, respectively. Using bisection method with (14) and (15), it is found that $b_{opt}=0.40651$ results in $(\sum_{k=0}^{19} c_k^2)_{\max}=0.243$, which is very close to the ideal value $(1/2\pi)\int_{2\pi}|\tilde{H}(\omega)|^2\, d\omega=0.25$. From (7), we see that $\varepsilon_{\min}(b)=0.007$. The corresponding magnitude and phase responses of Laguerre LPF in comparision with FIR filter (which is designed by *rectangular window* method [3]) are shown in Fig. 2. It is seen that the Laguerre response has appropriate linear phase and lower ripple in stopband compared to FIR filter response.

3. NUMERICAL METHOD

If $x(n)$ is a sequence of length N and $X(K)$, $\overline{X}(\omega)$ and $X(z)$ are its corresponding discrete fourier transform (DFT), fourier transform (FT) and Z-transform (ZT), respectively, it is known that [4]

$$X(K) = \overline{X}(\omega_K) = X(z)|_{z=e^{j\omega_K}} \qquad (16)$$

where $\omega_K=K2\pi/N$ and $K=0,1,\cdots,N-1$. If $\tilde{h}(n)$ is the unit-sample response of a linear time-invariant system with time span length of N, for $x(n)=\tilde{h}(N-n)$ as input in (4), it is shown [1,2] that $c_k=v_k(N)$. From (4), (16) and $\mathcal{Z}\{x(n)\}=\mathcal{Z}\{\tilde{h}(N-n)\}=z^{-N}\tilde{H}(z^{-1})$, for an $L+1$ stages Laguerre network, we can write

$$\begin{aligned} V_k(K) &= V_k(z)|_{z=e^{j\omega_K}} = [L_k(z,b)\,X(z)]|_{z=e^{j\omega_K}} \\ &= |\tilde{H}(\omega_K)|\, e^{-j\eta_K}\, L_k(e^{j\omega_K},b) \qquad (17) \end{aligned}$$

where $k=0,1,\cdots,L$, $K=0,1,\cdots,N-1$; $|\tilde{H}(\omega_K)|$, and η_K are the magnitude and phase responses of ideal filter $\tilde{H}(\omega)$ at ω_K, respectively. From (1), after some algebraic manipulations we obtain

$$\begin{aligned} L_k(e^{j\omega_K}) &= \sqrt{\frac{1-b^2}{1+b^2-2b\cos\omega_K}} \qquad (18) \\ &\exp\{j[K\pi/N-(1+2k)\theta_K]\} \end{aligned}$$

where $\theta_K=\arctan[(1+b)\tan(K\pi/N)/(1-b)]$ and ω_K is as in (16). Inserting (18) in (17), results in

$$\begin{aligned} V_k(K) &= |\tilde{H}(\omega_K)|\sqrt{\frac{1-b^2}{1+b^2-2b\cos\omega_K}} \qquad (19) \\ &\exp\{j[K\pi/N-(\eta_K+(1+2k)\theta_K)]\} \end{aligned}$$

From the identity of inverse discrete fourier transform (IDFT), we can write

$$c_k(b) = v_k(N) = IDFT\{V_k(K)\}|_{n=0} = \frac{1}{N}\sum_{K=0}^{N-1} V_k(K) \qquad (20)$$

Considering the above results, the procedure for designing the $L+1$ stages Laguerre filter is as follows:

i) Using *exhaustive* search method, increase b gradually from -1 to +1.
ii) Using (19) and (20), to compute corresponding $c_k(b)$'s for each b, and obtain the frequency response as: $H(\omega)=\sum_{k=0}^{L} c_k(b)\, L_k(e^{j\omega}, b)$.
iii) For each set of $\{c_k\}$ in step (ii), compute the MSE, $E(b)=(\sum_{K=0}^{N-1} |\tilde{H}(\omega_K) - H(\omega_K)|^2)/N$.
iv) Obtaining the optimum Laguerre filter parameters b_{opt} and $\{c_k\}_{opt}$, whenever $E(b)$ in step (iii) is minimum.

Now, we repeat example 1, by this method.

Example 2

We design a linear phase LPF with $\omega_c = \pi/4$ and $\tau=16$. To hold the linearity of phase, we take $N=33$ samples from $\tilde{H}(\omega)$ in $[0, 2\pi]$. Thus, we have

$$|\tilde{H}(\omega_K)| = \begin{cases} 1 & 0 \le K \le 4 \quad , \quad 29 \le K \le 32 \\ 0 & 5 \le K \le 28 \end{cases} \tag{21}$$

and

$$\eta_K = \begin{cases} -K\frac{32\pi}{33} & 0 \le K \le 16 \\ -(K-33)\frac{32\pi}{33} & 17 \le K \le 32 \end{cases} \tag{22}$$

Applying the numerical procedure, it yields a Laguerre network with 20 stages ($L=19$) with $b_{opt}=0.32$ and $E_{\min}=0.0026$. The corresponding magnitude and phase responses of the Laguerre LPF in comparision to FIR filter (which is designed by *frequency sampling* technique [3]) are shown in Fig. 3. It is seen that the response of the Laguerre filter has linear phase in passband, and lower ripple in stopband compared to FIR filter response.

4. FREQUENCY TRANSFORMATION

We assume that for linear phase LPF with ω_{cLP} and τ, the $L+1$ stages Laguerre network realization has optimum parameters as b_{Lopt} and $\{c_{kLP}\}_{opt}$. Now, we want to design a high-pass Laguerre digital filter with the same L and τ, but with cutoff frequency

$$\omega_{cHP} = \pi - \omega_{cLP} \tag{23}$$

From (15) and (23), it can be readily seen that $s(n)_{HP} = (-1)^{n-\tau}\, s(n)_{LP}$. Now, suppose that b_L and $b_H=-b_L$ are corresponding Laguerre parameters for LP and HP filters, respectively. Thus, from (14), after some algebraic manipulations we obtain

$$c_{kHP}(b_H) = (-1)^{k+\tau}\, c_{kLP}(b_L) \tag{24}$$

Consequently, if b_{Lopt} is optimum Laguerre parameter for LPF, then $b_{Hopt}=-b_{Lopt}$ is the corresponding optimum for HPF, provided that (23) is satisfied. Also, (24) shows that, in this case $\{c_{kHP}\}_{opt} = (-1)^{k+\tau}\, \{c_{kLP}\}_{opt}$.

5. CONCLUSIONS

Most of the FIR filter design methods have high ripples in the passband and stopband and a large number of *delay lines* (z^{-1}) in their realization. To improve this problem, more complicated approches have already been introduced in literature [3]. The simple algorithm introduced in this paper shows that the Laguerre digital filters have appropriate *linear phase* , *lower ripples* in passband and stopband and lower stages compared to FIR filters.

APPENDIX A

To show that

$$\sum_{k=0}^{L} c_k(b)\, \frac{d}{db} c_k(b) = \frac{L+1}{1-b^2}\, c_L(b)\, c_{L+1}(b) \tag{A.1}$$

we use the *induction* method. We show that (A.1) is true for $L=0$ (equivalent to one stage Laguerre network). Then, we assume that (A.1) is true for L terms, and demonstrate that it is true for $L+1$ terms.

But, first we show the validity of the following identity

$$\begin{aligned} L_{k+1}(z^{-1}, b) &= \frac{k}{k+1} L_{k-1}(z^{-1}, b) \\ &+ \frac{1-b^2}{k+1} \frac{d}{db} L_k(z^{-1}, b) \end{aligned} \tag{A.2}$$

Recalling (1), constracting appropriate $L_i(z^{-1}, b)$'s and substituting them in both sides of (A.2), it can be readily seen that (A.2) is valid.

Using (10) and (A.2), we obtain

$$c_{k+1}(b) = \frac{k}{k+1}\, c_{k-1}(b) + \frac{1-b^2}{k+1}\, \frac{d}{db}\, c_k(b) \tag{A.3}$$

From (1), it can be easily shown that

$$\frac{d}{db} L_0(z^{-1}, b) = \frac{1}{1-b^2}\, L_1(z^{-1}, b) \tag{A.4}$$

Thus, analogous to (A.3), from (10) and (A.4), we can easily show the validity of (A.1) for $L=0$.

Now, we suppose that the following identity is true

$$\sum_{k=0}^{L-1} c_k(b)\, \frac{d}{db} c_k(b) = \frac{L}{1-b^2}\, c_{L-1}(b)\, c_L(b) \tag{A.5}$$

and then we will show that (A.1) is true. We expand the left hand side of (A.1) as

$$\sum_{k=0}^{L} c_k(b)\,\frac{d}{db}c_k(b) = \sum_{k=0}^{L-1} c_k(b)\,\frac{d}{db}c_k(b) + c_L(b)\,\frac{d}{db}c_L(b) \quad (A.6)$$

Substituting (A.5) in (A.6), leads to

$$\sum_{k=0}^{L} c_k(b)\,\frac{d}{db}c_k(b) = \frac{L+1}{1-b^2}\,c_L(b) \left[\frac{L}{L+1}\,c_{L-1}(b) + \frac{1-b^2}{L+1}\,\frac{d}{db}c_L(b)\right] \quad (A.7)$$

Now if we set $k = L$ in (A.3), we see that the term in bracket in (A.7) is $c_{L+1}(b)$. Thus, (A.1) is verified.

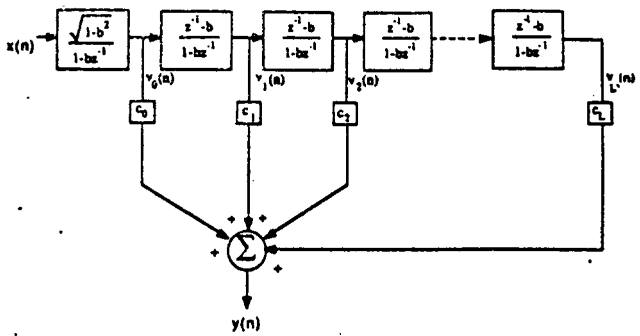

Fig. 1. Discrete-time Laguerre network.

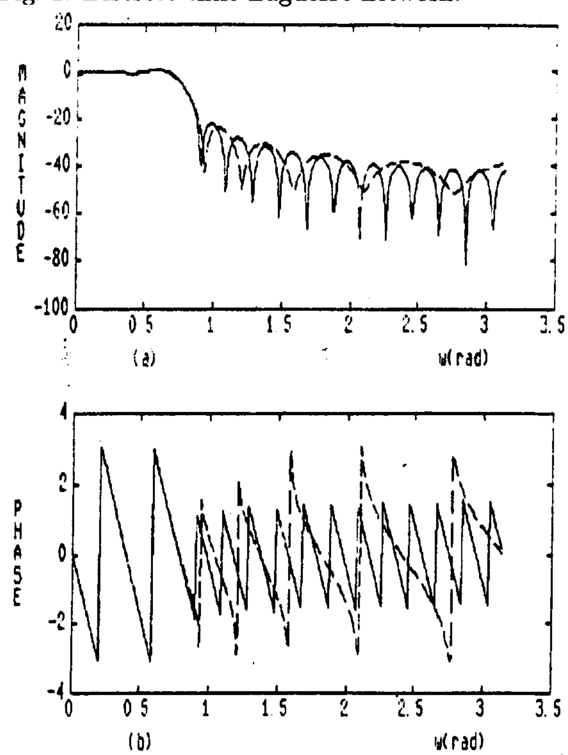

Fig. 2. FIR and Laguerre(dashed line) LPF (a)magnitude(db) and (b)phase(rad) response for analytical method.

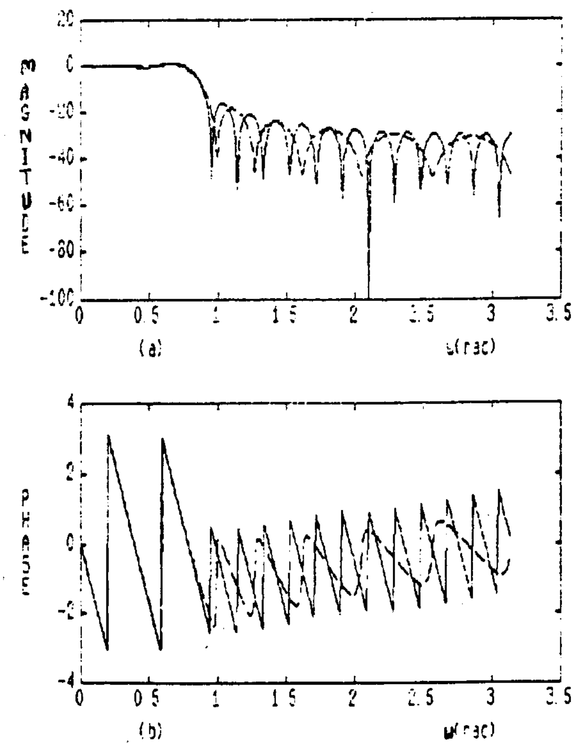

Fig. 3. FIR and Laguerre(dashed line) LPF (a)magnitude(db) and (b)phase(rad) response for numerical method.

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