# A NEW WLS METHOD FOR THE DESIGN OF TWO-DIMENSIONAL FIR DIGITAL FILTERS

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## ABSTRACT

Recently, the Weighted Least Square (WLS) technique for the FIR filter design has received wide attention, since it is computationally more efficient than other minimax techniques. However, for two-dimensional (2-D) filter design , the conventional WLS technique rearranges the frequency samples and the impulse filter coefficients with 2-D form into 1-D form, thus the WLS technique results in expensive computation. This paper presents a new 2-D FIR filter design method which retains the frequency samples and impulse filter coefficients in their 2-D form. Experimental results show that the new technique is computationally very efficient and leads to nearly-optimal approximations.

## 1. INTRODUCTION

In recent years, several techniques for the design of two-dimensional (2-D) FIR filters have been investigated. Much effort has been spent on designing filters which are optimal in the minimax (Chebyshev) sense. The minimax design yields an equiripple filter and requires the use of sophisticated optimization tools such as the linear programming [1], Remez exchange algorithm [2] or frequency transformation algorithm [3]. Kamp and Thiran [4] extended the Remez exchange algorithm for one-dimension (1-D) to the design of 2-D FIR filters. The optimal solution is not necessarily unique and may fail to converge. Another method for designing equiripple FIR filter is developed by McClellan [3], which transforms a 1-D Chebyshev design into a 2-D design through a change of variables. This method is limited that not all magnitude functions can be closely approximated. Recently, the weighted least squares (WLS) method [5]-[7] has received wide attention because it is computationally more efficient and the solution can be obtained analytically.

The WLS technique in [5] has extended 1-D filter design by using a formal relation of WLS to Chebyshev approximation and by exploiting this relation to develop an algorithm modifing Lawson's algorithm [8] for the design of 2-D filters.

To our best knowledge, so far the design of 2-D filters using WLS technique is based on the direct extension of the one dimension by rearranging the 2-D frequency samples and the impulse filter coefficients into a 1-D form. This paper presents a novel technique which retains the frequency samples and impulse filter coefficients in 2-D form. The new technique reduces the computation from $O(N^6)$ to $O(N^3)$, where N is the filter order. Moreover, we also present a scheme for the updation of the weighting function to improve the convergent speed.

## 2. DESIGN OF 2-D FILTER BY WLS METHOD [6][8]

In this section, we will review the conventional 2-D FIR filter design using the WLS technique. The frequency response of a 2-D zero-phase FIR filter can be expressed as

$$H(\omega_1,\omega_2)=\sum_{n_1=-N}^{N}\sum_{n_2=-N}^{N}h(n_1,n_2)e^{-j(\omega_1 n_1+\omega_2 n_2)}, \qquad (1)$$

where $h(n_1,n_2)$ is the rectangularly sampled impuse response of the filter, $N$ is its order, and $\omega_1$ and $\omega_2$ are the horizontal and vertical frequencies respectively. In the following we use a quadrantal symmetry low-pass filter as an example to illustrate the conventional WLS technique. Using the symmetric property

$$h(n_1,n_2)=h(\pm n_1,\pm n_2), \qquad (2)$$

(1) can be generally expanded as

$$H(\omega_1,\omega_2)=\sum_{n_1=0}^{N}\sum_{n_2=0}^{N}a(n_1,n_2)\cos(\omega_1 n_1)\cos(\omega_2 n_2) \qquad (3)$$
$$=\sum_{i=1}^{F}a(i)\varphi_i(\omega_1,\omega_2)$$

where $a(i)$ represents $a(n_1,n_2)$ which is associated with the impulse response samples $h(n_1,n_2)$ , $i$ is the function of $(n_1,n_2)$, and $F=(N+1)^2$ is the number of free filter coefficients. The $a(i)$ and $\varphi_i(\omega_1,\omega_2)$ can be readily derived respectively as follows :

$$a(n_1,n_2)=\begin{cases}h(0,0), & \text{for } n_1=0 \text{ and } n_2=0\\ 2h(n_1,n_2), & \text{for } n_1=0, n_2\neq 0 \text{ or } n_2=0, n_1\neq 0\\ 4h(n_1,n_2), & \text{otherwise}\end{cases} \qquad (4)$$

$$\varphi_i(\omega_1,\omega_2)=\cos(\omega_1 n_1)\cos(\omega_2 n_2). \qquad (5)$$

The ideal frequency response is defined as

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$$\hat{H}(\omega_1,\omega_2)=\begin{cases}1, \text{for } (\omega_1,\omega_2) \text{ in passband region} \\ 0, \text{for } (\omega_1,\omega_2) \text{ in stopband region,}\end{cases} \quad (6)$$

and varies linearly between 1 and 0 in the transition band. With the specification stated above the error can be written as

$$E(\omega_1^j,\omega_2^k)=\sum_i a(i)\varphi_i(\omega_1^j,\omega_2^k)-\hat{H}(\omega_1^j,\omega_2^k) \quad (7)$$

where $(L+1)\times(L+1)$ rectangular grid is chosen for the evaluation of the error in the first quadrant of $(\omega_1,\omega_2)$-plane. Then (7) can be rewritten as a vector-matrix form : $\mathbf{E}=\Phi\mathbf{a}-\hat{\mathbf{H}}$ , whereas the parameters are defined as

$$\mathbf{E}=\left[E(\omega_1^0,\omega_2^0), E(\omega_1^0,\omega_2^1), \cdots E(\omega_1^1,\omega_2^0), E(\omega_1^1,\omega_2^1), \cdots E(\omega_1^L,\omega_2^L)\right]^T$$

$$\mathbf{a}=\left[a(1),a(2),\cdots,a(F)\right]^T$$

$$\hat{\mathbf{H}}=\left[\hat{H}(\omega_1^0,\omega_2^0), \hat{H}(\omega_1^0,\omega_2^1), \cdots \hat{H}(\omega_1^1,\omega_2^0), \cdots \hat{H}(\omega_1^L,\omega_2^L)\right]^T.$$

$$\Phi=\begin{bmatrix}\varphi_1(\omega_1^0,\omega_2^0) & \cdots & \varphi_2(\omega_1^0,\omega_2^0) & \cdots & \varphi_F(\omega_1^0,\omega_2^0) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \varphi_1(\omega_1^1,\omega_2^0) & \ddots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ \varphi_1(\omega_1^L,\omega_2^L) & \cdots & \varphi_2(\omega_1^L,\omega_2^L) & \cdots & \varphi_F(\omega_1^L,\omega_2^L)\end{bmatrix}.$$

The optimal impulse filter coefficients have the form $\mathbf{a}=\left(\Phi^T R\Phi\right)^{-1}\Phi^T R\hat{\mathbf{H}}$ , whereas the diagonal weighting matrix is readily extended by

$$R=\begin{bmatrix}r(\omega_1^0,\omega_2^0) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & 0 \\ \vdots & 0 & r(\omega_1^1,\omega_2^0) & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & r(\omega_1^L,\omega_2^L)\end{bmatrix}.$$

The weighting function can be calculated in an iterative way such as [5]

$$r_{k+1}(\omega_1^i,\omega_2^j)=\frac{\left[r_k(\omega_1^i,\omega_2^j)+\lambda\right]\left|E_k(\omega_1^i,\omega_2^j)\right|}{\sum_m\sum_n\left[r_k(\omega_1^m,\omega_2^n)+\lambda\right]\left|E_k(\omega_1^m,\omega_2^n)\right|} \quad (8)$$

where the small positive constant $\lambda$ is utilized to avoid the propagation of zero weight.

It is clear that the 2-D filter design is basically a direct extension of the 1-D filter design, so we refer to such technique as a 1-D WLS for convenience.

The major computational burden for solving $\mathbf{a}$ lies in the computation of the inverse of the matrix $\Phi^T R\Phi$ with dimension $F^2\times F^2=(N+1)^2\times(N+1)^2$ .

## 3. PROPOSED METHOD FOR 2-D FIR FILTER DESIGN

Without loss of generality, we use again the example of the 2-D filter mentioned in Section 2. Recall that (3) can be expressed in a matrix form as

$$H=PAP^T \quad (9)$$

where

$$H=\left[H(i,j)\right]=\left[H(\frac{i\pi}{L},\frac{j\pi}{L})\right], \quad 0\le i,j\le L \quad (10)$$

$$P=\begin{bmatrix}\cos(0\times\frac{\pi}{L}\times 0) & \cos(0\times\frac{\pi}{L}\times 1) & \cdots & \cos(0\times\frac{\pi}{L}\times N) \\ \cos(1\times\frac{\pi}{L}\times 0) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ \cos(L\times\frac{\pi}{L}\times 0) & \cdots & \ddots & \cos(L\times\frac{\pi}{L}\times N)\end{bmatrix} \quad (11)$$

$$A=\begin{bmatrix}a(0,0) & \cdots & a(0,N) \\ \vdots & \ddots & \vdots \\ a(N,0) & \cdots & a(N,N)\end{bmatrix} \quad (12)$$

The desired frequency response is given by the matrix

$$\hat{\mathbf{H}}=\left[\hat{H}(i,j)\right]=\left[\hat{H}(\frac{i\pi}{L},\frac{j\pi}{L})\right], \quad 0\le i,j\le L. \quad (13)$$

The error matrix can be written as

$$\mathbf{E}=\left[E(i,j)\right]=\left[\hat{H}(\frac{i\pi}{L},\frac{j\pi}{L})-H(\frac{i\pi}{L},\frac{j\pi}{L})\right], \quad 0\le i,j\le L \quad (14)$$

We now define a new matrix $G=\left[G(i,j)\right]$ whose elements are given

$$G(i,j)=H(i,j)+\left(\frac{W(i,j)}{\alpha}\right)^{0.5}\times E(i,j)=H(i,j)+R(i,j)^{0.5}E(i,j) \quad (15)$$

where $R(i,j)=\frac{W(i,j)}{\alpha}$ is the weighting function corresponding to sampling frequency grid at $(\omega_1^i,\omega_2^j)=\left(\frac{i\pi}{L},\frac{j\pi}{L}\right)$ and $\alpha$ is a factor which affects the convergence and convergent speed. Thus the weighted square error to be minimized can be written as

$$\sum_{i=0}^{L}\sum_{j=0}^{L}R(i,j)\left(E(i,j)\right)^2=tr\left[G^TG-2G^TPAP^T+\left(PAP^T\right)^T\left(PAP^T\right)\right]$$

As a result, the optimal solution of $A$ is given

$$A=\left(P^TP\right)^{-1}P^TGP\left(P^TP\right)^{-1} \quad (16)$$

It is obvious from (12) to (14) that the parameters ( $\mathbf{E}$, $A$ and $\hat{\mathbf{H}}$ ) are retained in 2-D form. Therefore we refer to this technique as 2-D WLS method.

The most complex part of the computation in (16) is the inversion of a matrix $P^TP$ with dimension $(N+1)\times(N+1)$ . Therefore the computational complexity is $O(N^3)$, which is much less than $O(N^6)$ required by the conventional 1-D WLS technique mentioned above.

The following weight updating procedure has been found experimentally to be applied in the proposed

WLS technique for 2-D FIR filters design.

$$W_{k+1}(i,j)=\frac{L^2\times\left[W_k(i,j)+\lambda\right]\left|E_k(i,j)\right|^{\theta}}{\sum_{m=0}^{L}\sum_{n=0}^{L}\left[W_k(m,n)+\lambda\right]\left|E_k(m,n)\right|^{\theta}} \quad (17)$$

where the constant $\theta$ affects the convergent speed. Our experience indicates that when $\theta = 1.2$ and $\alpha > 2$, the algorithm converges at a maximum speed.

## 4. DESIGN EXAMPLES

In this section, a circularly symmetric low-pass filter with different specifications is designed to evaluate the performance of the proposed technique. Results of our technique, in terms of CPU design time and peak ripple magnitude (prm), are compared with those obtained by using the 1-D WLS technique presented by Algazi et al. [5], and Gislason et al. [7] and the Harrs' ascent algorithm [9].

The designed filter size is $15 \times 15$, passband ($\omega_p$) and stopband ($\omega_s$) radius are $0.4\pi$ and $0.6\pi$, respectively. The ratio of maximum error in passband and stopband is equal to one. Fig. 1 shows our algorithm converges rapidly to achieve approximately the same peak ripple magnitude. A comparison of the results obtained by the proposed algorithm and the Gislasons' algorithm [7] is shown in Table. 1 . This table depicts the design time and the peak ripple magnitude in the frequency response with different specifications and orders under the same sampling density $\left(L/_{N-1}=10\right)$. It is clear that in each case the peak ripple magnitude is very small and comparable to that obtained by using Gislasons' algorithm. The most important is that the design time using the proposed technique is significantly lower than that using the Gislasons' algorithm. Table 2 shows a comparison of Algazi's algorithm [5] with the proposed technique in the design of the low-pass filter with different specifications, orders, and sampling density $\left(128/_{N-1}\right)$. It is obvious that the peak ripple magnitude obtained by our technique is superior to that obtained by using the Algazi's algorithm.

Harris and Mersereau [9] have shown a comparison of several optimal (in minimax sense) 2-D FIR filter design techniques based on ascent algorithms. It turns out that Harris' algorithm converges faster than other ascent algorithms. A comparison between the Harris' algorithm and proposed algorithm based on the same sampling density $\left(L/_{N-1}\right)$ is shown in Table 3. It is worth noting that, as shown in Fig. 2 , the rate of increase in design time for the proposed technique is significantly smaller than that for Harris' algorithm, although the two algorithms are implemented on different computers. Therefore, higher order filters can be designed more efficiently by using the proposed technique .

## 5. CONCLUSION

The paper has presented a new 2-D FIR filter design using the 2-D WLS technique. The new technique retains the frequency samples and the impulse filter coefficients in a 2-D form. The design time needed is proportional to $N^3$ rather than $N^6$ required by the conventional WLS. Experimental results have demonstrated that the proposed technique is quite efficient and approximately leads to a minimax design in two dimensions.

## REFERENCES


[1] L. R. Rabiner, "The design of finite impulse response digital filters using linear programming techniques," Bell Syst, Tech, J., vol. 51, pp.1117-1198, 1972.

[2] T. W. Parks, and J. H. McClellan, "Chebyshev approximation for nonrecursive digital filters with linear phase," IEEE Trans. Circuit Theory, vol. CT-19, pp. 189-194, 1972.

[3] J. H. McClellan, "The design of 2-D digital filters by transformations," in Proc. 7th Annu. Princeton Conf. Information Sciences and Systems, pp.247-251, 1973.

[4] Y. Kamp, J. P. Thirm, "Chebyshev approximation for 2-D nonrecursive digital filters," IEEE Trans. CAS,Vol. CAS-22, pp.208-218, 1975.

[5] V. R. Algazi, et al., "Design of almost minimax FIR filters in one and two dimensions by WLS techniques," IEEE Trans. CAS, Vol CAS-33, pp.590-596, June 1986.

[6] Y. C. Lim, et al., "A weightted least square algorithm for quasi-equiripple FIR and IIR digital filter design," IEEE Trans. SP, Vol. 40, pp.551-558, . 1992.

[7] Gislason, et al., "Three different criteria for the design of 2-D zero phase FIR digital filters,"IEEE Trans. SP, Vol. 41, pp.3070-3074, 1993.

[8] C. L. Lawson, "Contribution to the theory of linear least maximum approximation," Ph.D. dissertation, U.C.L.A., 1961.

[9] D. B. Harris and R. M. Mersereau, "A comparison of algorithms for minimax design of linear FIR digital filters," IEEE Trans. ASSP, Vol. ASSP-25, pp. 492-500, 1977.

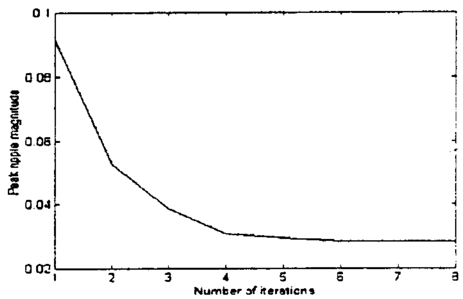


Fig. 1 Convergent speed of the proposed algorithm

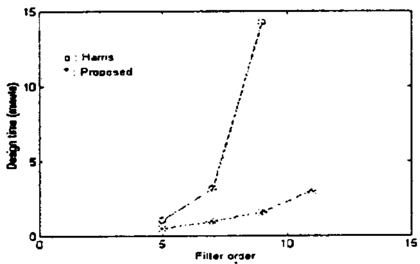


Fig.2 Design time for circularly symmetric low-pass filter.

Table 1 A comparison between Gislasons' [7] and the proposed algorithm.

| $\omega_p$ | $\omega_s$ | N | Algorithm | prm | time (iteration) |
|---|---|---|---|---|---|
| 0.45π | 0.55π | 5 | Gislason | 0.3657 | 0:51 (12 ) |
| | | | Proposed | 0.3733 | 0:07 ( 7 ) |
| | | 7 | Gislason | 0.2355 | 3:35 (13 ) |
| | | | Proposed | 0.2413 | 0:24 (11 ) |
| | | 9 | Gislason | 0.2331 | 13:40 (18 ) |
| | | | Proposed | 0.2358 | 0:48 (11) |
| 0.4π | 0.6π | 5 | Gislason | 0.2549 | 0:45 (10 ) |
| | | | Proposed | 0.2609 | 0:11 (11 ) |
| | | 7 | Gislason | 0.1206 | 3:03 (10 ) |
| | | | Proposed | 0.1266 | 0:22 (10 ) |
| | | 9 | Gislason | 0.1134 | 10:01 (10 ) |
| | | | Proposed | 0.1146 | 0:32 ( 8 ) |
| 0.35π | 0.65π | 5 | Gislason | 0.1830 | 0:37 ( 8 ) |
| | | | Proposed | 0.1869 | 0:11 (11 ) |
| | | 7 | Gislason | 0.0664 | 3:15 (11 ) |
| | | | Proposed | 0.0637 | 0:16 (7 ) |
| | | 9 | Gislason | 0.0561 | 9:31 (9) |
| | | | Proposed | 0.0551 | 0:47 (12) |

Table 2 A comparison between Algazi's [5] and the proposed algorithm.(d=128/N-1 )

| $\omega_p$ | $\omega_s$ | N | Algorithm | d | prm | time (iteration) |
|---|---|---|---|---|---|---|
| 0.45π | 0.55π | 5 | Algazi | 32 | 0.3755 | 0:35( 6 ) |
| | | | Proposed | 32 | 0.3719 | 0:20( 6 ) |
| | | 7 | Algazi | 21 | 0.2529 | 0:41( 7 ) |
| | | | Proposed | 21 | 0.2505 | 0:21( 6 ) |
| | | 9 | Algazi | 16 | 0.2341 | 1:00( 10 ) |
| | | | Proposed | 16 | 0.2293 | 0:30( 10 ) |
| 0.4π | 0.6π | 5 | Algazi | 32 | 0.2733 | 0:41( 7 ) |
| | | | Proposed | 32 | 0.2670 | 0:32( 11 ) |
| | | 7 | Algazi | 21 | 0.1336 | 0:53( 9 ) |
| | | | Proposed | 21 | 0.1254 | 0:28( 9 ) |
| | | 9 | Algazi | 16 | 0.1202 | 0:50( 8 ) |
| | | | Proposed | 16 | 0.1131 | 0:26( 8 ) |
| 0.35π | 0.65π | 5 | Algazi | 32 | 0.1916 | 0:40( 7 ) |
| | | | Proposed | 32 | 0.2016 | 0:28( 9 ) |
| | | 7 | Algazi | 21 | 0.0650 | 0:52( 9 ) |
| | | | Proposed | 21 | 0.0639 | 0:23( 7 ) |
| | | 9 | Algazi | 16 | 0.0579 | 0:30( 5 ) |
| | | | Proposed | 16 | 0.0560 | 0:40( 14 ) |

*(VAX 11 / 780 for Lawson's and IBM PC 486-50 for Proposed )

Table 3 A comparison between Harris' [9] and the proposed algorithm.(d=L/N-1)

| $\omega_p$ | $\omega_s$ | N | Algorithm | d | prm | time (iteration) |
|---|---|---|---|---|---|---|
| 0.45π | 0.55π | 5 | Harris | 15 | 0.3673 | 0:53 |
| | | | Proposed | 15 | 0.3696 | 0:17( 6 ) |
| | | 7 | Harris | 15 | 0.2473 | 3:03 |
| | | | Proposed | 15 | 0.2511 | 0:44( 7 ) |
| | | 9 | Harris | 15 | 0.2330 | 8:16 |
| | | | Proposed | 15 | 0.2365 | 1:46(10) |
| 0.4π | 0.6π | 5 | Harris | 20 | 0.2670 | 1:18 |
| | | | Proposed | 20 | 0.2650 | 0:28(11) |
| | | 7 | Harris | 20 | 0.1272 | 3:10 |
| | | | Proposed | 20 | 0.1286 | 0:64(11) |
| | | 9 | Harris | 20 | 0.1141 | 14:14 |
| | | | Proposed | 20 | 0.1169 | 1:36( 9 ) |
| 0.35π | 0.65π | 5 | Harris | 15 | 0.1905 | 1:06 |
| | | | Proposed | 15 | 0.2123 | 0:26(10 ) |
| | | 7 | Harris | 15 | 0.0656 | 1:42 |
| | | | Proposed | 15 | 0.0664 | 0:49( 8 ) |
| | | 9 | Harris | 15 | 0.0557 | 8:11 |
| | | | Proposed | 15 | 0.0564 | 2:22(14) |

*( PDP 11 / 50 for Harris' and IBM PC 486-50 for Proposed )