WEIGHTED LEAST-SQUARES DESIGN OF LINEAR-PHASE AND ARBITRARY 2-D COMPLEX FIR FILTERS

Amin G. Jaffer*, William E. Jones* and Theagenis J. Abatzoglou**

*Hughes Aircraft Company
Fullerton, California USA
**Lockheed Research Laboratories
Palo Alto, California USA

ABSTRACT

This paper presents the design of 2-D complex FIR filters using the weighted integral least-squares error criterion (WLS). Both the cases of arbitrary magnitude with linear and arbitrary phase specifications are addressed. The solution of the linear phase case is obtained using the complex Lagrange multiplier formulation to incorporate the necessary constraints for linear phase response. This results in a computationally efficient filter design technique requiring the solution of a Hermitian Toeplitz-block-Toeplitz system of linear equations for which fast algorithms are available. Two illustrative filter design examples are also presented.

1. INTRODUCTION

The design of 1-D complex FIR filters which satisfy specified asymmetric amplitude or phase responses necessary in radar/sonar clutter suppression problems and other applications, has been considered recently [1], [2]. More recently, the problem of complex 2-D finite impulse response (FIR) filter design has been investigated recently by some authors [3], [4]. The work of [3] has focused on the use of eigenfilter techniques which however only approximately minimize the WLS criterion and also require more computation than the direct WLS optimization methods presented here. The work presented in [4] is related to the present paper although the method described there is not as general since it is restricted to even length filters.

Two WLS techniques, one for arbitrary phase response (unconstrained method) and the other incorporating the linear phase constraint (constrained method), are developed here. The direct WLS optimization methods developed here utilize the complex gradient operator [5] which avoids decomposing complex variables into real and imaginary parts. Due to the product dimensionality of the 2-D complex coefficient vector, the computational aspects become of paramount importance. The methods presented here are computationally efficient for general complex 2-D filter design. Furthermore, this paper makes use of a 2-D factorization theorem developed in a companion paper [6] to further reduce the computational demands for a certain class of filters. Additionally, for a further special but useful class of 2-D filters, we show that our techniques result in a solution which altogether avoids the need for matrix inversion or the solution of a linear system of equations thus reducing the computational demands dramatically. The relationship between the constrained and unconstrained techniques is also examined. Finally, two illustrative filter design examples are presented with comparison with the results in [3].

2. WEIGHTED LEAST-SQUARES COMPLEX 2-D FIR FILTER DESIGN

We derive here the solutions for the complex 2-D FIR filter coefficients which minimize the weighted integral least-squares error criterion both subject to affine (generalized linear) phase constraints as well as being unconstrained. The latter design would be suitable for arbitrarily specified magnitude and phase requirements. Let $z_D(f_1,f_2)=a_D(f_1,f_2)\,e^{j\phi_D(f_1,f_2)}$ be the desired complex-valued 2-D frequency response where $a_D(f_1,f_2)$ and $\phi_D(f_1,f_2)$ represent the amplitude and phase responses respectively. Let

$$H(f_1,f_2)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}h_{m,n}\,e^{-j2\pi mf_1}\,e^{-j2\pi nf_2} \quad (1)$$

be the frequency response of a 2-D complex FIR filter with coefficients $h_{m,n}$ using a normalized sampling frequency of 1 Hertz. Let $\underline{h}$ be the $(MN\times 1)$ concatenated vector that "vectorizes" the 2-D filter coefficient matrix with elements $h_{m,n}$, i.e. the vector that consists of the stacked columns of the aforementioned matrix. We seek to minimize the weighted integral squared-error criterion

$$J(\underline{h})=\int_0^1\int_0^1 w(f_1,f_2)\left|\,z_D(f_1,f_2)-\underline{d}_K^H(f_1,f_2)\,\underline{h}\,\right|^2 df_1\,df_2 \quad (2)$$

where $\underline{d}_K(f_1,f_2)=\underline{d}_1(f_1)\otimes\underline{d}_2(f_2)$, ($\otimes$ denotes Kronecker product of matrices [7])

$$\underline{d}_1(f_1)=\left[1, e^{j2\pi f_1},\ldots,e^{j2\pi(M-1)f_1}\right]^T$$

$$\underline{d}_2(f_2)=\left[1, e^{j2\pi f_2},\ldots,e^{j2\pi(N-1)f_2}\right]^T$$

and $w(f_1,f_2)$ is a real non-negative weighting function. The superscripts H and T denote the conjugate transpose and transpose operations respectively. It is noted that (2) allows for incorporation of zero-error 2-D transition bands by letting $w(f_1,f_2)$ be zero over those regions.

2.1 Constrained Weighted Least-Squares Method

The necessary and sufficient conditions for a 2-D complex FIR filter to have affine phase (generalized linear phase with an offset) response has been derived by Abatzoglou [4] using an extension of the method developed in [2] for the 1-D case. Using the linear phase (zero offset) conditions results in the filter coefficients being constrained to be conjugate-symmetric,

$$h_{m,n} = h^*_{M-1-m,N-1-n} \quad \text{for all } m,n \tag{3}$$

where * denotes complex conjugation.

The solution is obtained by appending the constraints (3) to (2) using complex Lagrange multipliers. A solution has been obtained by Abatzoglou in [4] by directly incorporating the constraints in the objective function (2) which is, however, valid only for even product length MN. The solution given here using Lagrange multipliers is more general and valid for both even and odd product lengths MN.

The constraints (3) can be compactly represented as $\underline{h}^* = E_{MN}\,\underline{h}$ where $\underline{h}$ is the concatenated $(MN\times 1)$ vector of 2-D filter coefficients and E_{MN} is the MN-order square exchange matrix with ones on the cross-diagonal and zeros elsewhere. Since the objective function augmented with the constraints using the complex Lagrange multipliers must be real, we have

$$\begin{aligned} J_1(\underline{h}) = J(\underline{h}) - \underline{\lambda}^T\left[\underline{h}^* - E_{MN}\,\underline{h}\right] \\ - \underline{\lambda}^H\left[\underline{h} - E_{MN}\,\underline{h}^*\right] \end{aligned} \tag{4}$$

where $\underline{\lambda}$ is a complex Lagrange multiplier vector. Expanding (4) and differentiating with respect to $\underline{h}^*$ according to the rules of the complex partial derivatives [5] and equating to the null vector yields:

$$\begin{aligned} \frac{\partial J_1(\underline{h})}{\partial \underline{h}^*} = \int_0^1\!\!\int_0^1 \big(-w(f_1,f_2)\, z_D(f_1,f_2)\, \underline{d}_K(f_1,f_2) \\ + w(f_1,f_2)\, \underline{d}_K(f_1,f_2)\, \underline{d}_K^H(f_1,f_2)\underline{h}\big)\, df_1\, df_2 \\ - \underline{\lambda} + E_{MN}\,\underline{\lambda}^* = \underline{0} \end{aligned} \tag{5}$$

(Note: The exchange matrix E_{MN} satisfies $E_{MN}^T = E_{MN}$ and $E_{MN}^2 = I_{MN}$ where I_{MN} is the identity matrix).

Let

$$Q = \int_0^1\!\!\int_0^1 w(f_1,f_2)\,\underline{d}_K(f_1,f_2)\,\underline{d}_K^H(f_1,f_2)\,df_1\,df_2 \tag{6}$$

$$\underline{u} = \int_0^1\!\!\int_0^1 w(f_1,f_2)\,z_D(f_1,f_2)\,\underline{d}_K(f_1,f_2)\,df_1\,df_2 \tag{7}$$

Note that Q is a Hermitian Toeplitz-block-Toeplitz or doubly-Toeplitz matrix. From (5), (6) and (7), one obtains

$$Q\,\underline{h} - \underline{u} = \underline{\lambda} - E_{MN}\,\underline{\lambda}^*$$

Let

$$\underline{\gamma} = \underline{\lambda} - E_{MN}\,\underline{\lambda}^*$$

then

$$Q\,\underline{h} = \underline{u} + \underline{\gamma} \tag{8}$$

We next determine the $\underline{\gamma}$ vector so that the constraint $\underline{h}^* = E_{MN}\,\underline{h}$ is satisfied. From (8), we have

$$E_{MN}\,Q\,\underline{h} = E_{MN}\,\underline{u} + E_{MN}\,\underline{\gamma} \tag{9}$$

Since Q is Hermitian Toeplitz-block-Toeplitz and, in particular, Hermitian-persymmetric we have that [8]

$$E_{MN}\,Q = Q^*\,E_{MN} \tag{10}$$

Also, $$E_{MN}\,\underline{\gamma} = E_{MN}\,\underline{\lambda} - \underline{\lambda}^* = -\underline{\gamma}^* \tag{11}$$

From (9), (10) and (11), one has

$$Q^*\,E_{MN}\,\underline{h} = E_{MN}\,\underline{u} - \underline{\gamma}^*$$

and from (8)

$$Q^*\,\underline{h}^* = \underline{u}^* + \underline{\gamma}^*$$

and, hence, by subtraction

$$Q^*\left[\underline{h}^* - E_{MN}\,\underline{h}\right] = 2\,\underline{\gamma}^* + \underline{u}^* - E_{MN}\,\underline{u} \tag{12}$$

The use of the constraint $\underline{h}^* = E_{MN}\,\underline{h}$ in (12) yields

$$\underline{\gamma} = \frac{1}{2}\left[E_{MN}\underline{u}^* - \underline{u}\right]$$

and, from (8), the solution of the 2-D filter coefficient vector as

$$\begin{aligned} Q\,\underline{h} = \frac{1}{2}\left[\underline{u} + E_{MN}\,\underline{u}^*\right] \\ \text{or} \quad \underline{h} = \frac{1}{2}Q^{-1}\left[\underline{u} + E_{MN}\,\underline{u}^*\right] \end{aligned} \tag{13}$$

It can be readily verified that the solution (13) does indeed satisfy the conjugate symmetry constraints.

Since Q is Hermitian Toeplitz-block-Toeplitz, (13) can be solved using fast algorithms as discussed under "Remarks." We also note here that (13) can be expressed in an alternate form for even product length fillers (where MN is even). Since $\underline{h}$ is conjugate symmetric, it can be written in terms of the upper half coefficient vector $\underline{h}_u$ as

$$\underline{h} = \begin{bmatrix} \underline{h}_u \\ E_{MN/2}\,\underline{h}_u^* \end{bmatrix}$$

and (13) can be written as

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^H & Q_{22} \end{bmatrix} \begin{bmatrix} \underline{h}_u \\ E_{MN/2}\,\underline{h}_u^* \end{bmatrix} = \frac{1}{2}\left\{\begin{bmatrix} \underline{u}_p \\ \underline{u}_L \end{bmatrix} + \begin{bmatrix} E_{MN/2}\,\underline{u}_L^* \\ E_{MN/2}\,\underline{u}_p^* \end{bmatrix}\right\} \tag{14}$$

where $Q_{22} = Q_{11}$ and $\underline{u}$ has been similarly partitioned into $\underline{u}_p$ and $\underline{u}_L$. Using well-known matrix inversion techniques for a 2×2 partitioned block matrix, the solution for the upper half coefficient vector $\underline{h}_u$ can be expressed as

$$\begin{aligned} \underline{h}_u = \frac{1}{2}\left[Q_{11} - Q_{12}\,Q_{11}^{-1}\,Q_{12}^H\right]^{-1} \\ \left[\underline{u}_p + E_{MN/2}\,\underline{u}_L^* - Q_{12}\,Q_{22}^{-1}\left(\underline{u}_L + E_{MN/2}\,\underline{u}_p^*\right)\right] \end{aligned} \tag{15}$$

Although the dimensionality of (15) is half that of (13), the expression for $\underline{h}_u$ is more complicated requiring additional products of matrices and inversion of a matrix $\left[Q_{11} - Q_{12}\,Q_{11}^{-1}\,Q_{12}^H\right]$ which is not necessarily block Toeplitz.

2.2 Unconstrained 2-D Filter Design Methods

The solution is obtained immediately from the preceding development in Section 2.1 by deleting the constraints and Lagrange multipliers to yield

$$Q\underline{h}=\underline{u} \quad \text{or} \quad \underline{h}=Q^{-1}\underline{u} \qquad (16)$$

Of course, equation (16) does not in general satisfy the conjugate symmetry constraints.

2.3 Remarks

1. The solutions (13) or (16) for the 2-D filter coefficient vector require solving a set of MN simultaneous linear equations whose associated matrix is Toeplitz-block-Toeplitz. Fast algorithms exist for inversion of such matrices [9] requiring $o(\min(M,N)M^2N^2)$ complex flops as opposed to $o(M^3N^3)$ complex flops for general matrix inversion techniques resulting in a computational reduction by a factor of $\max(M,N)$. As noted in Section 2.4, special cases of 2-D filter design allow even further computational reduction. Additionally, if the weighting function is taken as uniform over the entire f_1-f_2 plane, the Q matrix reduces to the identity matrix as can be seen from the following:

The matrix $\underline{d}_k(f_1,f_2)\underline{d}_k^H(f_1,f_2)$ is Hermitian Toeplitz-block-Toeplitz whose first "block" row is

$$[\ \underline{d}_2(f_2)\underline{d}_2^H(f_2),\ e^{-j2\pi f_1}\underline{d}_2(f_2)\underline{d}_2^H(f_2),\ \ldots,\ e^{-j2\pi(M-1)f_1}\underline{d}_2(f_2)\underline{d}_2^H(f_2)\ ] \qquad (17)$$

Since the $(m,n)th$ element of $\int_0^1\underline{d}_2(f_2)\underline{d}_2^H(f_2)\,df_2$ is $\int_0^1 e^{j2\pi(m-n)f_2}\,df_2=1$, for $m=n$ and zero otherwise, integration of (17) with respect to f_2 yields a "block" row

$$[I_N,\ e^{-j2\pi f_1}I_N,\ \ldots,\ e^{-j2\pi(M-1)f_1}I_N] \qquad (18)$$

Further integration of (18) with respect to f_1 yields the first "block" row of Q as

$$[\ I_N,\ \underline{0},\ \ldots,\ \underline{0}\ ]$$

and hence, using the Hermitian-block-Toeplitz property, we have $Q=I_{MN}$.

This property allows certain special cases of the 2-D filter design to be obtained in a computationally trivial manner since no solution of a system of linear equations is required. Although zero-error transition regions corresponding to zero weighting are not allowed in this formulation, other suitable response models in the transition bands using uniform weighting may be substituted.

2. Although the filter coefficient vector resulting from (13) satisfies the conjugate symmetry constraint whereas that from (14) does not, in general, it is worth noting that if the desired response phase function $\phi_D(f_1,f_2)$ is taken as linear phase with delays $\tau_1=(M-1)/2$ and $\tau_2=(N-1)/2$, then, surprisingly, the unconstrained solution yields the same solution as the constrained one. This was shown for the 1-D case in [2] and can be seen for the 2-D case from the following:

With $z_D(f)=a_D(f_1,f_2)\,e^{-j2\pi f_1(M-1)/2}\,e^{-j2\pi f_2(N-1)/2}$

$$\underline{u}=\int_0^1\int_0^1 w(f_1,f_2)\,a_D(f_1,f_2)\left[e^{-j2\pi f_1(M-1)/2}\,\underline{d}_1(f_1)\right]\otimes\left[e^{-j2\pi f_2(N-1)/2}\,\underline{d}_2(f_2)\right]df_1\,df_2 \qquad (19)$$

Since $E_{MN}=E_M\otimes E_N$, we have

$$E_{MN}\,\underline{u}^*=\int_0^1\int_0^1 w(f_1,f_2)\,a_D(f_1,f_2)\left[e^{j2\pi f_1(M-1)/2}\,E_M\,\underline{d}_1^*(f_1)\right]\otimes\left[e^{j2\pi f_2(N-1)/2}\,E_N\,\underline{d}_2^*(f_2)\right]df_1\,df_2 \qquad (20)$$

It can be seen that the two expressions (19) and (20) are the same and hence equations (13) and (14) become the same for this case.

2.4 2-D Factorization Theorem

It has been shown in [6] that if the 2-D weighting and desired response functions $w(f_1,f_2)$ and $z_D(f_1,f_2)$ are factorable in terms of their corresponding 1-D functions over rectangular regions in the f_1-f_2 plane, i.e.

$$w(f_1,f_2)=w_1(f_1)w_2(f_2)$$

and

$$z_D(f_1,f_2)=z_{D1}(f_1)z_{D2}(f_2)$$

then the 2-D filter coefficient vector is "factorable," i.e. it can be expressed as

$$\underline{h}=\underline{h}_1\otimes\underline{h}_2 \qquad (21)$$

where $\underline{h}_1$, $\underline{h}_2$ are the corresponding 1-D filter coefficients of lengths M and N respectively where

$$\underline{h}_1=\frac{1}{2}Q_1^{-1}\left[\underline{u}_1+E_M\,\underline{u}_1^*\right] \quad \text{or} \quad \underline{h}_1=Q_1^{-1}\underline{u}_1$$

as appropriate for the constrained and unconstrained cases. Q_1, $\underline{u}_1$ are the corresponding 1-D values [2]. Similarly,

$$\underline{h}_2=\frac{1}{2}Q_2^{-1}\left[\underline{u}_2+E_N\,\underline{u}_2^*\right] \quad \text{or} \quad \underline{h}_2=Q_2^{-1}\underline{u}_2$$

for the f_2 filter.

Equation (21) implies a substantial computational savings capability over the general 2-D filter design, for designing a certain class of 2-D filters. Note that zero-error transition bands defined over rectangular regions in the f_1-f_2 plane certainly satisfy the condition for weighting function factorability since $w(f_1,f_2)=0$ on those regions.

A further property of semi-factorability is noted here. Even if the desired response function is non-factorable but the weighting function is, then the 2-D filter coefficient vector can be computed as[6]

$$\underline{h}=\left[Q_1^{-1}\otimes Q_2^{-1}\right]\underline{u}$$

where the inverses required are $(M\times M)$ and $(N\times N)$ as opposed to the full 2-D filter design method requiring an $(MN\times MN)$ matrix inversion.

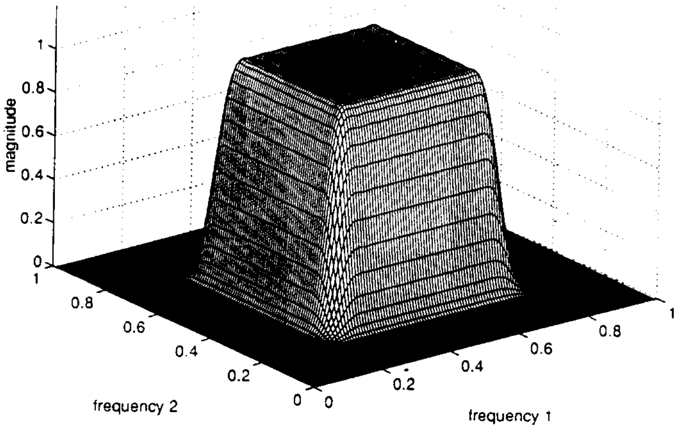

Figure 1 – 2-D Rectangular Filter Response.

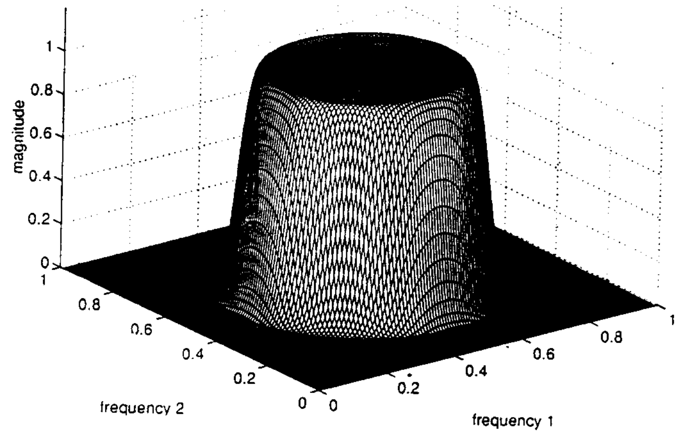

Figure 2 – 2-D Circular Filter Response.

3.0 COMPUTER DESIGN EXAMPLES

Two illustrative filter design examples, low-pass rectangular and circular 2-D filters, are presented here. While, in general, the evaluation of the Q matrix and the $\underline{u}$ vector require two-dimensional numerical integration, in many cases these quantities can be computed in closed form such as for rectangular 2-D filters with piecewise linear amplitude response specifications and uniform weighting functions. Also, in some cases such as the circular 2-D filter example presented below, partial closed-form formulas can be used to reduce computation. In both examples below, the desired phase response is $\phi_D(f_1, f_2) = e^{-j2\pi f_1(M-1)/2} e^{-j2\pi f_2(N-1)/2}$

Rectangular Low-Pass 2-D Filter Design. Figure 1 shows the 2-D frequency response for a 2-D linear phase 24×24 rectangular filter designed using two 1-D filters of lengths 24 each utilizing the method described in Section 2.4 (the 2-D Factorization Theorem). The passband extends from .3 to .7 in both f_1 and f_2 dimensions while the stopband extends from 0 to .2 and .8 to 1 in both the f_1 and f_2 dimensions. The remaining unspecified regions constitute the transition band. The peak passband and stopband errors are .005743 (.049 dB) and .002876(-50.82dB) as contrasted to peak errors of .005826(.051 dB) and .003607 (-48.86dB) obtained in [3]. The design time was 6 seconds on a Macintosh Quadra 700.

Circular Low-Pass 2-D Filter Design. Figure 2 shows the 2-D frequency response for a 2-D linear phase 25×25 circular filter which could not be factored into two 1-D filters. The two dimensional integral required for the computation of Q and $\underline{u}$ defined over a circular region is computed using a combination of a closed-form formula for the inner integral and numerical integration using Simpson's rule. The circular passband radius is .25 centered at frequency $f_1 = f_2 = .5$. The transition region is the circular annulus between radii .25 and .35. The remaining region constitutes the stopband. The peak passband and stopband errors are .00525 (.045 dB) and .0092 (-40.73dB) as contrasted to peak errors of .0068(.059 dB) and .007398(-42.6 dB) obtained in [3].

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