A CLOSED-FORM LEAST SQUARES SOLUTION TO THE DISCRETE FREQUENCY DOMAIN DESIGN PROBLEM OF TWO-DIMENSIONAL FIR FILTERS

Magdy T. Hanna
Department of Electrical Engineering
University of Bahrain
P.O. Box 32038
Isa Town
Bahrain

ABSTRACT

Based on a discrete frequency domain formulation of the design problem of two-dimensional FIR filters, a closed form expression is derived for the matrix of filter coefficients without imposing any assumptions of having a symmetric, antisymmetric or zero-phase frequency response. The matrix in question is derived by minimizing the Frobenius norm of the difference between the matrices of the actual and ideal frequency responses at the points of a frequency grid. The method has the advantages of conceptual and computational simplicity.

I. INTRODUCTION

Ahmad and Wang presented an analytic closed-form solution to the least squares design problem of real impulse response (with a square region of support) zero-phase 2-D FIR filters with a quadrantally symmetric or antisymmetric frequency response [1]. They formulated the problem as a discrete frequency domain optimization one and used the properties of matrices and trigonometric functions. Recently Rajaravivarma and Rajan derived a closed form solution to the problem using time-domain optimization and employing the DFT transformation matrices [2]. They first presented the solution to the general frequency response specifications and then simplified it for the cases of quadrantal symmetry and antisymmetry.

In this paper a discrete frequency domain formulation based on matrix notation is presented for the general case where no assumptions of a real impulse response or zero-phase frequency response are made. A closed-form expression will be derived for the impulse response matrix.

II. MATHEMATICAL FRAMEWORK

The frequency response of a two-dimensional FIR filter is given by [3] :

$$H(\omega_1,\omega_2)=\sum_{n_1=-N_1}^{N_1}\sum_{n_2=-N_2}^{N_2} h(n_1,n_2)e^{-j(\omega_1 n_1+\omega_2 n_2)} \quad (1)$$

Although the impulse response has a rectangular region of support centered at the origin, there is no implied assumption of symmetry or antisymmetry, i.e., $H(\omega_1,\omega_2)$ is not generally zero-phase. Moreover $h(n_1,n_2)$ is not restricted to be real. Equation (1) can be compactly expressed as[1] :

$$H(\omega_1,\omega_2)=f^+(\omega_1)Ag(\omega_2) \quad (2)$$

where A is the $(2N_1+1)\times(2N_2+1)$ matrix :

$$A=\begin{bmatrix} h(-N_1,-N_2) & \cdots & h(-N_1,N_2) \\ \cdot & \cdots & \cdot \\ \cdot & \cdot\cdot & \cdot \\ h(N_1,-N_2) & \cdot\cdot & h(N_1,N_2) \end{bmatrix} \quad (3)$$

and $f(\omega_1)$ and $g(\omega_2)$ are respectively $(2N_1+1)$- and $(2N_2+1)$-dimensional vectors defined by :

$$f(\omega_1)=\begin{bmatrix} Ju^*(\omega_1) \\ 1 \\ u(\omega_1) \end{bmatrix} \text{ and } g(\omega_2)=\begin{bmatrix} Jv^*(\omega_2) \\ 1 \\ v(\omega_2) \end{bmatrix}. \quad (4)$$

In the above equation $u(\omega_1)$ and $v(\omega_2)$ are respectively N_1- and N_2-dimensional vectors defined by[2] :

$$u(\omega_1)=\begin{pmatrix} e^{j\omega_1} \\ \cdot \\ \cdot \\ e^{jN_1\omega_1} \end{pmatrix} \text{ and } v(\omega_2)=\begin{pmatrix} e^{-j\omega_2} \\ \cdot \\ \cdot \\ e^{-jN_2\omega_2} \end{pmatrix} \quad (5)$$

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[1]The superscript + , * , T denotes the complex conjugate transpose, the complex conjugate and the transpose respectively.

[2]Notice that the exponents of the elements of vectors $u(\omega_1)$ and $v(\omega_2)$ are positive and negative respectively.

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and J is the contra-identity matrix (having ones on the antidiagonal and zeros elsewhere) of the proper order .
It is straightforward to show that :

$$u^*(\omega_1) = u(-\omega_1) \tag{6}$$

$$f^*(\omega_1) = Jf(\omega_1) \tag{7}$$

and similar properties hold for the vectors $v(\omega_2)$ and $g(\omega_2)$.
Discretizing the continuous frequencies ω_1 and ω_2 by taking $2M_1$ and $2M_2$ samples of them respectively, we get :

$$\left(\omega_i\right)_{m_i} = -\pi + \frac{\pi}{M_i}\left(m_i - 1\right) \qquad , m_i = 1, \ldots, 2M_i \qquad , i=1,2 \tag{8}$$

It will be assumed that the number of frequency samples in each direction is greater than or equal to the length of the impulse response in that direction, i.e., $2M_i \geq (2N_i + 1)$ for i = 1,2 . Defining the scalars $H_{m1,m2}$ as samples of the frequency response $H(\omega_1,\omega_2)$ - of the filter to be designed - at the above discrete frequencies, i.e.,

$$H_{m_1,m_2} = H\left(\left(\omega_1\right)_{m_1}, \left(\omega_2\right)_{m_2}\right) \tag{9}$$

and using Eq(2), we get :

$$H_{m_1,m_2} = f^+_{m_1} A g_{m_2} \tag{10}$$

where the vectors f_{m1} and g_{m2} are given by :

$$f_{m_1} = f\left(\left(\omega_1\right)_{m_1}\right) \qquad , m_1 = 1, \ldots, 2M_1 \tag{11}$$

$$g_{m_2} = g\left(\left(\omega_2\right)_{m_2}\right) \qquad , m_2 = 1, \ldots, 2M_2 \tag{12}$$

Let $\underline{H}$ be the $2M_1 \times 2M_2$ matrix of the frequency response samples $H_{m1,m2}$, i.e.,

$$\underline{H} = \left[H_{m_1,m_2} \right] . \tag{13}$$

Using Eq(10), the above matrix can be expressed as :

$$\underline{H} = F^+ A G \tag{14}$$

where F and G are respectively the $(2N_1 + 1) \times 2M_1$ and $(2N_2 + 1) \times 2M_2$ matrices defined by :

$$F = \left(f_1 \ \cdots \ f_{2M_1} \right) \tag{15}$$

$$G = \left(g_1 \ \cdots \ g_{2M_2} \right) . \tag{16}$$

III. THE OPTIMIZATION PROBLEM

Let $\underline{H}°$ be the matrix of samples of the ideal (desired) frequency response at the same discrete frequencies of Eq (8). The square of the Frobenius norm [4] of the matrix difference $\underline{H} - \underline{H}°$ is defined by :

$$E = \left\|\underline{H} - \underline{H}°\right\|^2 = \sum_{m_1=1}^{2M_1} \sum_{m_2=1}^{2M_2} \left| H_{m_1,m_2} - H°_{m_1,m_2} \right|^2 \tag{17}$$

and can also be expressed as[3] :

$$E = tr\left[\left(\underline{H} - \underline{H}°\right)^+ \left(\underline{H} - \underline{H}°\right) \right] . \tag{18}$$

Using Eq(14) and the properties of the trace of a matrix it can be shown that :

$$E = tr\Big(GG^+A^+FF^+A - A^+FH°G^+ - GH°^+F^+A + H°^+H° \Big) \tag{19}$$

The rectangular matrix A of the impulse response of the FIR filter to be designed will be derived by minimizing the above error function. Here use will be made of the following theorem whose proof is given in the appendix :

<u>Theorem</u> :

The minimizer of the function :

$$f(A) = tr\Big(BA^+CA - A^+D - D^+A \Big) \tag{20}$$

where B and C are nonsingular Hermitian matrices of order n and m respectively and A and D are m x n complex matrices, is given by :

$$A = C^{-1}DB^{-1} . \tag{21}$$

Applying the above result to Eq(19), we get :

$$A = \left(FF^+\right)^{-1} FH°G^+ \left(GG^+\right)^{-1} . \tag{22}$$

Using the properties of the pertinent matrices and performing a substantial algebraic manipulation, it can be proved that :

$$FF^+ = 2M_1 I . \tag{23}$$

and

$$GG^+ = 2M_2 I . \tag{24}$$

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[3]tr(A) denotes the trace of matrix A.

Substituting Eqs (23) and (24) into (22), one gets the following compact expression for matrix A of the impulse response of the FIR filter being designed :

$$A = \frac{1}{4M_1M_2} FH^\circ G^+ . \tag{25}$$

APPENDIX

The Theorem of Section III will be proved by stating and proving the following lemmas :

Lemma 1 :

If A is a real m x n matrix and B and C are square real matrices of order n and m respectively, then :

$$\nabla_A tr\left(BA^TCA\right) = CAB + C^TAB^T . \tag{A1}$$

Proof : Let the columns of matrix A be denoted by a_i and the rows of matrix B by $b^{(i)}$, i.e.,

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} \tag{A2}$$

$$B = \begin{pmatrix} b^{(1)} \\ b^{(2)} \\ \\ \cdot \\ b^{(n)} \end{pmatrix} \tag{A3}$$

and let e_i be the i th column vector of the identity matrix. Therefore :

$$\begin{aligned} tr\left(BA^TCA\right) &= \sum_{i=1}^{n} e_i^T\left(BA^TCA\right)e_i \\ &= \sum_{i=1}^{n} b^{(i)}A^TCa_i \end{aligned} \tag{A4}$$

Since

$$A^TCa_i = \begin{pmatrix} a_1^TCa_i \\ \cdot \\ \cdot \\ a_n^TCa_i \end{pmatrix} \tag{A5}$$

Eq(A4) can be expressed as :

$$f \equiv tr\left(BA^TCA\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}a_j^TCa_i \tag{A6}$$

Since

$$\nabla_x\left(y^Tx\right) = \nabla_x\left(x^Ty\right) = y \tag{A7}$$

and

$$\nabla_x\left(x^TCx\right) = \left(C + C^T\right)x \tag{A8}$$

the gradient of the function of Eq(A6) with respect to the column vector a_k is given by :

$$\begin{aligned} \nabla_{a_k} f &= \sum_{\substack{i=1 \\ i\neq k}}^{n} b_{ik}Ca_i + \sum_{\substack{j=1 \\ j\neq k}}^{n} b_{kj}C^Ta_j + b_{kk}\left(C + C^T\right)a_k \\ &= \sum_{i=1}^{n} b_{ik}Ca_i + \sum_{j=1}^{n} b_{kj}C^Ta_j \\ &= C\sum_{i=1}^{n} a_ib_{ik} + C^T\sum_{j=1}^{n} a_jb_{kj} \\ &= CAb_k + C^TA\left[b^{(k)}\right]^T \end{aligned} \tag{A9}$$

Defining the gradient of f with respect to matrix A as :

$$\nabla_A f = \begin{pmatrix} \nabla_{a_1} f & \nabla_{a_2} f & \dots & \nabla_{a_n} f \end{pmatrix} \tag{A10}$$

the validity of Eq(A1) is directly established .

Corollary 1 :

If the real square matrices B and C are both symmetric or skew-symmetric, then :

$$\nabla_A tr\left(BA^TCA\right) = 2CAB . \tag{A11}$$

Corollary 2 :

If one of the two real square matrices B and C is symmetric and the other is skew-symmetric, then :

$$\nabla_A tr\left(BA^TCA\right) = 0 . \tag{A12}$$

Lemma 2 :

If A and D are m x n real matrices, then :

$$\begin{aligned} &\nabla_A tr\left(D^TA\right) = \nabla_A tr\left(A^TD\right) = \\ &\nabla_A tr\left(DA^T\right) = \nabla_A tr\left(AD^T\right) = D \end{aligned} \tag{B1}$$

Proof :

Since

$$tr\left(D^TA\right) = \sum_{i=1}^{n} e_i^TD^TAe_i = \sum_{i=1}^{n} d_i^Ta_i \tag{B2}$$

applying (A7), one gets :

$$\nabla_{a_k} tr\left(D^T A\right) = d_k . \quad (B3)$$

Using definition (A10), one gets :

$$\nabla_A tr\left(D^T A\right) = D . \quad (B4)$$

The remaining parts of (B1) follow by the properties of the trace of a matrix .

Corollary :

If A and D are m x n real matrices and B is a real square matrix of order n, then :

$$\nabla_A tr\left(BA^T D\right) = DB \quad (B5)$$

Proof :

Eq (B5) follows directly from Eq (B1) since :

$$tr\left(BA^T D\right) = tr\left(DBA^T\right) \quad (B6)$$

Lemma 3 :

If A and D are m x n complex matrices and B and C are nonsingular Hermitian matrices of order n and m respectively, then the minimizer of :

$$f(A) = tr\left(BA^+ CA - A^+ D - D^+ A\right) \quad (C1)$$

is given by :

$$A = C^{-1} D B^{-1} . \quad (C2)$$

Proof :

Let the real and imaginary components of the relevant matrices be denoted by the subscripts r and i respectively, i.e.,

$$A = A_r + jA_i \quad (C3)$$

$$B = B_r + jB_i \quad (C4)$$

$$C = C_r + jC_i \quad (C5)$$

$$D = D_r + jD_i . \quad (C6)$$

Since B and C are Hermitian, B_r and C_r will be symmetric, and B_i and C_i will be skew-symmetric. Since

$$BA^+ = \left(B_r A_r^T + B_i A_i^T\right) + j\left(B_i A_r^T - B_r A_i^T\right) \quad (C7)$$

and

$$CA = \left(C_r A_r - C_i A_i\right) + j\left(C_r A_i + C_i A_r\right) \quad (C8)$$

it can be shown that :

$$\begin{aligned} BA^+CA = (&B_r A_r^T C_r A_r + B_i A_i^T C_r A_r - B_r A_r^T C_i A_i - B_i A_i^T C_i A_i \\ &+ B_r A_i^T C_r A_i + B_r A_i^T C_i A_r - B_i A_r^T C_r A_i - B_i A_r^T C_i A_r) \\ &+ j(B_r A_r^T C_r A_i + B_i A_i^T C_r A_i + B_r A_r^T C_i A_r + B_i A_i^T C_i A_r \\ &+ B_i A_r^T C_r A_r - B_i A_r^T C_i A_i - B_r A_i^T C_r A_r + B_r A_i^T C_i A_i) \end{aligned} \quad (C9)$$

Since

$$A^+ D = \left(A_r^T D_r + A_i^T D_i\right) + j\left(A_r^T D_i - A_i^T D_r\right) \quad (C10)$$

and

$$D^+ A = \left(D_r^T A_r + D_i^T A_i\right) - j\left(D_i^T A_r - D_r^T A_i\right) \quad (C11)$$

it can be shown that :

$$tr\left(A^+ D + D^+ A\right) = 2tr\left(A_r^T D_r + A_i^T D_i\right) . \quad (C12)$$

Substituting Eqs (C9) and (C12) into Eq (C1), taking the gradient of f with respect to A_r and A_i and utilizing Eqs (A11),A(12),(B1) and (B5), one gets :

$$\nabla_{A_r} f = 2\left(C_r A_r B_r - C_i A_r B_i - C_r A_i B_i - C_i A_i B_r - D_r\right) \quad (C13)$$

$$\nabla_{A_i} f = 2\left(C_r A_r B_i + C_i A_r B_r + C_r A_i B_r - C_i A_i B_i - D_i\right) . \quad (C14)$$

Applying the minimization conditions :

$$\nabla_{A_r} f = \nabla_{A_i} f = 0 \quad (C15)$$

and combining the resulting equations using Eqs (C3)-(C6), one gets after some algebraic manipulation :

$$CAB = D . \quad (C16)$$

Eq (C2) follows from the above equation by the nonsingularity of the matrices B and C .

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