ENFORCING A MINIMUM-PHASE CONDITION ON AN ARBITRARY ONE-DIMENSIONAL SIGNAL WITH APPLICATION TO A TWO-DIMENSIONAL PHASE RETRIEVAL PROBLEM

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ABSTRACT

In this paper, we consider the problem of making a minimum phase signal from an arbitrary one-dimensional signal by adding a point signal and its application to a two-dimensional phase retrieval problem. In particular, we show that a two-dimensional phase retrieval problem can be decomposed into several one-dimensional phase retrieval problems so that a $M \times N$ two-dimensional signal can be reconstructed from its Fourier transform magnitude by solving $\min\{M, N\} + 2$ one dimensional phase retrieval problems.

1. Introduction

The phase retrieval problem is a problem of reconstructing a signal or the Fourier transform phase of the signal from its Fourier transform magnitude [1]. This problem does not have a unique solution in general in the sense that there exist many signals that have the same Fourier transform magnitude [1]. In a one-dimensional phase retrieval problem, for example, the z-transform of any one-dimensional discrete-time sequence of length N has exactly $N-1$ zeros (i.e., factors), so that any finite length sequences obtained by flipping zeros have the same Fourier transform mangitude. However, if we have *à priori* information that the one-dimensional signal to be determined is minimum-phase (i.e., has all its zeros and poles inside the unit circle), then we can reconstruct the signal uniquely from its Fourier transform magnitude [2].

In this paper, we consider the problem of enforcing minimum-phase condition on an arbitrary one-dimensional signal by adding a delta function having a large amplitude. Then we consider the extension of the result to a two-dimensional phase retrieval problem.

2. Enforcing minimum-phase conditions on one-dimensional signals

2.1. Enforcing minimum-phase conditions

We firstly show that, if we add a delta function having a sufficiently large amplitude to an arbitrary one-dimensional signal, then the added signal can be a minimum-phase signal. We begin with the following theorem.

Theorem 1 *[3] Let $x(n)$ be an arbitrary real causal finite-support one-dimensional signal that has a minimum nonzero region of support $[0, N-1]$, i.e., $x(n) = 0$ if $n \notin [0, N-1]$ with $x(0) \neq 0$ and $x(N-1) \neq 0$, and $y(n)$ be a signal that is obtained by adding a point signal $A\delta(n)$ to the signal $x(n)$, i.e.,*

$$y(n) = x(n) + A\delta(n). \tag{1}$$

If $|A| > \sum_{n=0}^{N-1} |x(n)|$, then the signal $y(n)$ becomes a minimum-phase signal.

To show what Theorem 1 implies, we present an example.

Example I Figure 1.(a) is a 128 point original signal and Figure 1.(b) is the zero plot of the signal (a). As we can see in Figure 1.(b), some of the zeros are inside the unit circle and the others are outside the unit circle. Thus, this original signal is a mixed-phase signal (i.e., neither a minimum- phase nor a maximum-phase signal). Then, we add a delta function at n=0 whose amplitude is 100 which is greater than the summation of the absolute values of the coefficients, which is 57.1 in this case. Figure 1.(c) is the zero plot of the added signal. As we can see, all the zeros of this signal are inside the unit circle and therefore, this signal is a minimum phase signal. ◇

2.2. Reconstruction

When a Fourier transform magnitude is given, there are several ways to get the corresponding minimum-phase signal.

1. **Hilbert transform**

   First of all, we can reconstruct the signal using Hilbert transform [2]. It is shown that there is a Hilbert transform relationship between the real part and the imaginary part of a causal signal. Furthermore, the cepstrum of any minimum-phase signal is causal and therefore, the log magnitude of its Fourier transform magnitude and its phase have a Hilbert transform relationship [2].

2. **Root finding**

   The second way is using a root finding algorithm. Suppose we have a finite-length signal $x(n)$ having a minimum non-zero support $[0, N-1]$ and its z-transform $X(z)$ so that the zeros of $X(z)$ are given as $z_1, z_2, \cdots, z_{N-1}$. Then, $X(z)$ is given as

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$= \quad x(0)\prod_{i=1}^{N-1}(1-z_i z^{-1}).$$

From this equation, we can see that a signal can be uniquely determined from all its zeros and the first point $x(0)$.

Now suppose that we have Fourier intensity of an unknown minimum-phase signal $x(n)$. From the Fourier intensity (i.e., the square of the Fourier transform magnitude) we can obtain the autocorrelation function of the desired signal $x(n)$ which is given as

$$\begin{aligned} R_x(z) &= X(z)X^*(z^{-1}) \\ &= |x(0)|^2 \prod_{i=1}^{N-1}(1-z_i z^{-1})(1-z_i^* z). \end{aligned}$$

Thus, if $X(z)$ has a zero at $z = z_0$, then the z-transform of the autocorrelation function has two zeros at $z = z_0$ and the complex conjugate of its reciprocal $z = 1/z_0^*$. So, half of the zeros of $R_x(z)$ are outside the unit circle, and the other half are inside the unit circle, which are exactly the zeros of the minimum-phase signal. Furthermore, if the first point $x(0)$ is real and positive, this value can also be determined from the equation above.

3. **Gerchberg-Saxton algorithm**
   Another way is using a Gerchberg-Saxton algorithm. This algorithm has a very simple structure (Figure 2). When a phase retrieval problem is given, whether this algorithm converges to the desired signal or not has not been proved yet, however it has been shown that the error between the given Fourier transform magnitude and the Fourier transform magnitude of the estimated signal is not increasing [4]. And many experiments with the GS algorithm show that the algorithm has a tendency that for a phase retrieval problem having a unique solution, the estimated signal approaches the desired solution signal. Figure 3 is a comparison of the convergence properties of the mixed-phase signal and the minimum-phase signal in Example I after 10 iterations. As we can see in the figure, the Gerchberg-Saxton algorithm converges to the desired signal if the desired signal has a minimum-phase property.

3. Extension to a two-dimensional phase retrieval problem

3.1. Enforcing a minimum-phase condition

In this section, we consider the extension of the result in Section 2. to a two-dimensional phase retrieval problem. According to [5], a two-dimensional minimum-phase causal quadrant signal $b(m,n)$ is defined as a causal signal whose z-transform has the following property

$$B(z_1,z_2) \neq 0 \text{ for } \{|z_1| \geq 1, |z_2| \geq 1\}.$$

A multidimensional minimum-phase signal may be defined similarly.

First, we consider whether we can get a two-dimensional minimum-phase signal by adding a delta function as we did in Section 2.. We begin with the matrix

$$x = \begin{bmatrix} 1 & 2 & 10 & 4 \\ 2 & 3 & 4 & 5 \\ 10 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

Figure 4 is the zero contour of z_1 as z_2 moves along the unit circle in the z_2-plane (since the matrix is symmetric, we don't have to get the zero contour of z_2 along the unit circle of z_1.). As we can see in the picture, this signal is neither a minimum-phase signal nor a maximum-phase signal.

Then, we add a delta function having a large amplitude at the origin. Since the summation of all the absolute values of the signal is 77, we add 80, which is greater than the summation. Then, the added signal $y(m,n)$ is given as

$$y(m,n) = x(m,n) + 80\delta(m,n).$$

Figure 4(b) is the zero contour of the signal y. As we can see in the figure, all the zero contours are inside the unit circle and therefore the signal now became a minimum-phase signal.

We can theorize this phenomenon into the following theorem.

Theorem 2 *[3] Let $x(m,n)$ be an arbitrary two-dimensional signal that has the minimum nonzero support $[0, M-1] \times [0, N-1]$ and $y(m,n)$ be a two-dimensional signal that is obtained by adding a point signal $A\delta(m,n)$ to the signal $x(m,n)$, i.e.,*

$$y(m,n) = x(m,n) + A\delta(m,n)$$

If a positive real number A is given as

$$A > \sum_{m=0}^{M-1}\sum_{n=0}^{N-1}|x(m,n)|,$$

then the signal $y(m,n)$ become a minimum-phase signal and can be uniquely specified from its Fourier transform magnitude.

3.2. Reconstruction

The main idea of the reconstruction of the signal satisfying the condition in Theorem 2 is that the two-dimensional Fourier transform can be decomposed into the concatenation of the columnwise and the rowwise one-dimensional Fourier transforms so that the first row $y(m,0)$(for $m = 0,\cdots,M-1$) as well as all the columns $\bar{y}_k$'s (for $k = 0,1,\cdots,2M-1$) of the rowwise Fourier transform $y_k(n)$ (for $k = 0,\cdots,2M-1$) are minimum-phase by Theorem 1 [3], where $\bar{y}_k$ and $y_k(n)$ are given as

$$\bar{y}_k = [y_k(0), y_k(1), y_k(2), \cdots, y_k(2N-1)]^t,$$

and

$$y_k(n) = \sum_{m=0}^{2M-1} y(m,n)\exp\{-j\frac{\pi}{M}km\}.$$

Obviously, the columnwise Fourier transform of

$$[\bar{y}_0, \bar{y}_1, \cdots, \bar{y}_{2M-1}]$$

is $Y(k,l)$, the two-dimensional Fourier transform of $y(m,n)$.

The algorithm to reconstruct the desired signal from the given condition is as follows.

Algorithm

Step 1 Inverse Fourier transform columnwise all the columns of the Fourier intensity $|Y(k,l)|^2$ to get the autocorrelation function of $\bar{y}_k$ for $k = 0, \cdots, 2M-1$.

Step 2 Find all the zeros of each autocorrelation function that are inside the unit circle and $|y_k(0)|^2$.

Step 3 Find the minimum-phase signal that has $[|y_0(0)|^2, |y_1(0)|^2, \cdots, |y_{2M-1}(0)|^2]$ as its Fourier transform intensity.

Step 4 Fourier transform the minimum-phase signal to determine $y_k(0)$'s.

Step 5 For each column, form a polynomial columnwise using the zeros obtained in Step 2 and multiply $y_k(0)$ columnwise to make the signals $y_k(n)$.

Step 6 Inverse Fourier transform to get the desired signal.

The following example shows how the algorithm works.

Example II Let $y(m,n)$ be a 4×4 matrix that is obtained from $x(m,n)$ by adding $70\delta(m,n)$, where $x(m,n)$ is given as

$$x = \begin{bmatrix} 1 & 2 & 5 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 2 \end{bmatrix}, \quad y = \begin{bmatrix} 71 & 2 & 5 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 2 \end{bmatrix}$$

Obviously, the magnitude of the point signal 70 is greater than the summation of the all the absolute values of $x(m,n)$, which in this case is 61. Assume that we do not know the signal $y(m,n)$ but its Fourier transform magnitude, which is given as

$$\begin{bmatrix} 131 & 88 & 60 & 82 & 69 & 82 & 60 & 88 \\ 89 & 42 & 68 & 72 & 66 & 76 & 65 & 92 \\ 64 & 70 & 73 & 65 & 72 & 70 & 63 & 67 \\ 84 & 72 & 63 & 73 & 74 & 70 & 69 & 76 \\ 69 & 64 & 68 & 72 & 67 & 72 & 68 & 64 \\ 84 & 76 & 69 & 70 & 74 & 73 & 63 & 72 \\ 64 & 67 & 63 & 71 & 72 & 65 & 73 & 70 \\ 89 & 92 & 65 & 76 & 66 & 72 & 68 & 42 \end{bmatrix}$$

If we square the Fourier transform magnitude and inverse Fourier transform it columnwise, then we get the autocorrelation of $\bar{y}_k$'s. Since we know that all the $\bar{y}_k$'s are minimum-phase, the zeros of the z-transform of $\bar{y}_k$'s are inside the unit circle. The zeros of the z-transform of $\bar{y}_0$, $\bar{y}_1, \cdots$, $\bar{y}_4$ are as follows.

$$\left[\begin{matrix} -.5185 & .4485+.5217i & -.3095+.2460i \\ .1793+.6079i & .0162-.5121i & .3852+.0722i \\ .1793-.6059i & -.4911+.1257i & -.0464-.3494i \end{matrix} \right.$$

$$\left. \begin{matrix} .0961+.3692i & -.3672 \\ -.2605-.0881i & .1797+.2785i \\ .1175-.2578i & .1797-.2785i \end{matrix} \right]$$

Also, $|y_k(0)|^2$'s are given as

$$[6724, 4927, 4360, 5244, 4900, 5244, 4360, 4927]. \qquad (2)$$

Since these values are the Fourier intensity of the minimum-phase signal $y(m,0)$, we can find the zeros of the z-transform of $y(m,0)$ and are given as

$$\{-0.3306, 0.1512+0.3841i, 0.1512-0.3841i\}$$

as well as $y(0,0)$, which is 71. If we form the polynomial with the zeros and multiply it with $y(0,0)$, then we get the first row $y(m,0)$, which is given as $y(m,0) = 71 \times [1, .0282, .0704, .0563] = [71, 2, 5, 4]$. Performing Step 4 to Step 6, we get

$$\begin{bmatrix} 71 & 2 & 5 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 2 \end{bmatrix},$$

which is exactly the same as the original signal. ⋄

As we can see above, because of the symmetry we only need to solve $N+2$ one-dimensional phase retrieval problems ($N+1$ columns and $y(m,0)$), where N is the number of columns. If we switch columns and rows in the algorithm, we can also get the same results with $M+2$ problems, where M is the number of rows. Thus, we can solve the same two-dimensional phase retrieval problem by solving $\min\{M,N\}+2$ one-dimensional phase retrieval problems.

4. Conclusion

In this paper, we considered the problem of enforcing minimum-phase conditions on arbitrary one-dimensional signals and its application to phase retrieval problem. Then, we extended this results to two-dimensional signals such that we can make any two-dimensional signal have minimum-phase property and, when we apply this to a phase retrieval problem, we can solve the problem by solving several one-dimensional phase retrieval problems.

In solving a phase retrieval problem, the off-axis holography technique is a robust method in the sense that it guarantees the unique solution all the time. However, this method requires at least two times bigger support than that of a typical phase retrieval problem for each dimension. For a two-dimensional phase retrieval problem, for example, at least 2×2 times bigger support is required than that of a typical phase retrieval problem. The method presented here, however, can not only solve the phase retrieval with the same support as that of a typical phase retrieval problem but also can be implemented by the same technique as that of the off-axis holography technique.

5. References

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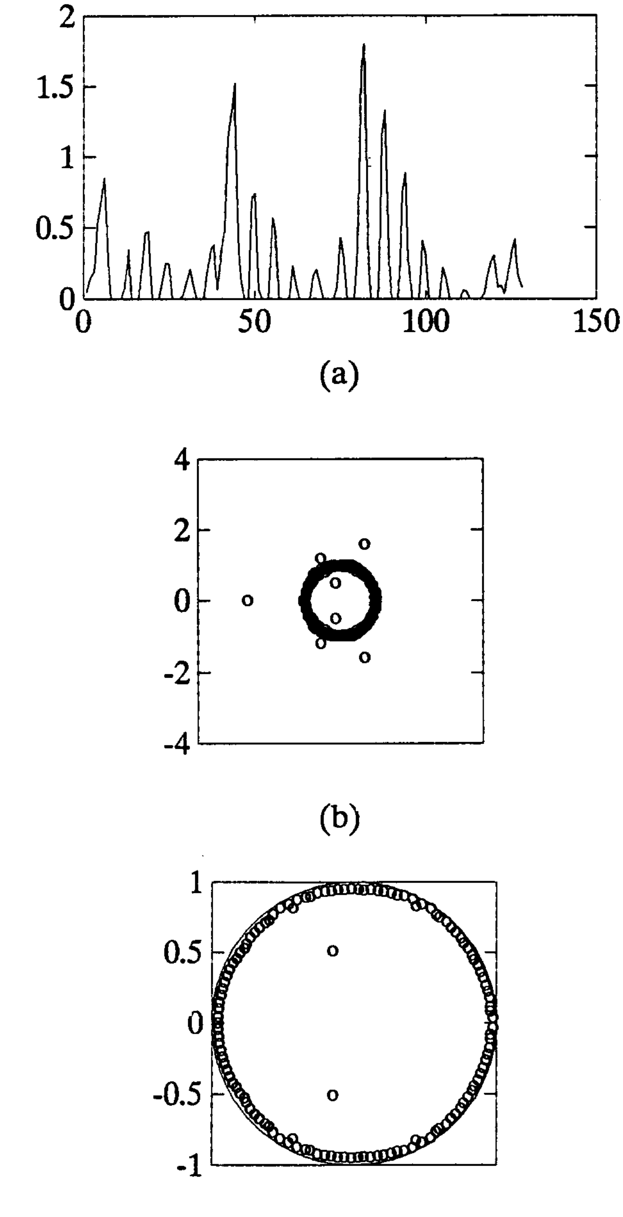

Figure 1: Zero plot of a 128-point one-dimensional signal; (a). a 128-point original signal, (b). Zero plot of the signal (a), (c). Zero plot of the signal added by a delta function.

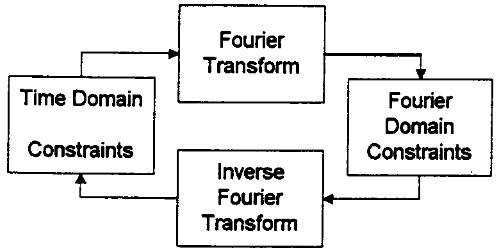

Figure 2: Gerchberg-Saxton Algorithm

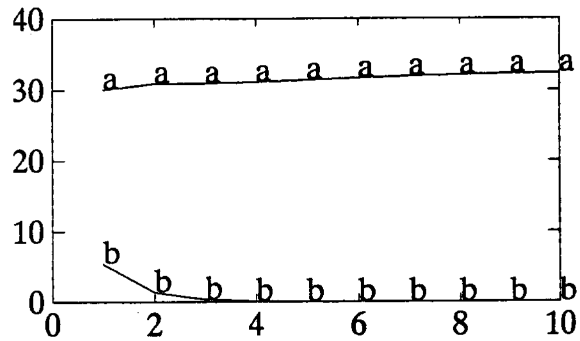

Figure 3: Convergence property of a mixed-phase signal (marked as 'a') and minimum-phase signal (marked as 'b') using the Gerchberg-Saxton algorithm

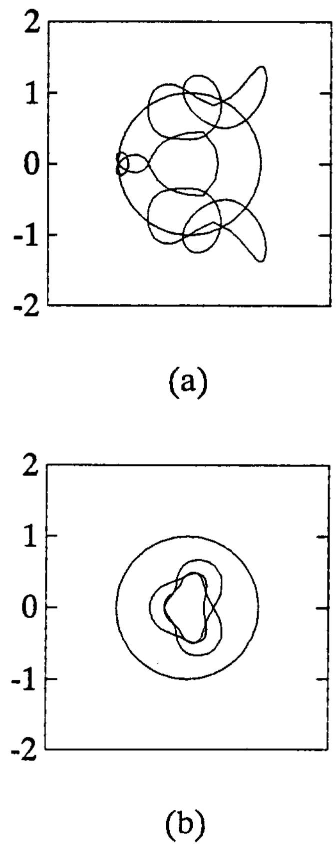

Figure 4: Zero-contour plot of (a) a two-dimensional signal (mixed-phase) and (b) the signal added by a delta function at the origin (minimum-phase)