REGULARIZED EXTRAPOLATION OF NOISY DATA WITH A WAVELET SIGNAL MODEL

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ABSTRACT

We examine a regularization technique for robust data extrapolation based on the wavelet representation in this research. We first formulate the regularization problem and characterize the properties of its solution. Then, a practical iterative algorithm is proposed to achieve robust extrapolation.

1. INTRODUCTION

The band-limited signal model has been widely used in the past three decades [8], and band-limited extrapolation has been extensively studied and applied in signal reconstruction [7]. Possible applications of signal extrapolation include spectrum estimation, synthetic aperture radar (SAR) imaging, limited-angle tomography, beamforming and high resolution image restoration. The performance of an extrapolation algorithm is highly dependent on a proper modeling of the underlying signal. There are however signals which are not band-limited such as nonstationary signals. Wavelet theory has recently attracted a lot of attention as a useful tool for signal modeling, and the multiresolution wavelet representation leads naturally to a scale-limited signal model. To illustrate the additional modeling power of the scale-limited model, we may consider the following two examples. First, the cubic cardinal B-spline wavelet basis [1] spans a function space whose elements are second-order polynomials between knots and with continuous first-order derivate at knots. Many practical signals can be well approximated with such a function space. Second, time-localized wavelet bases such as the Haar and Daubeches wavelets are more suitable than the conventional Fourier basis in modeling signals with interesting transient information such as those arising from the electrocardiogram and radar applications.

A new signal extrapolation technique based on the wavelet representation, known as scale/time-limited extrapolation, was studied by Xia, Kuo and Zhang [11]. However, the extrapolated result may be unstable due to the ill-posedness of the extrapolation problem, we examine a regularization technique for robust data extrapolation in this research.

2. PROBLEM FORMULATION

The scale-limited signal model is based on multiresolution analysis and wavelet theory. Consider a sequence of successive approximation space $\mathcal{P}_j$ of $L^2(\mathbf{R})$ satisfying,

$$\cdots \subset \mathcal{P}_{-2} \subset \mathcal{P}_{-1} \subset \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \cdots,$$

with

$$\overline{\bigcup_j \mathcal{P}_j} = L^2(\mathbf{R}), \qquad \bigcap_j \mathcal{P}_j = \{0\}.$$

Let $\phi(t)$ be the associated scaling function and we define $\phi_{jk}(t) = 2^{j/2}\phi(2^j t - k)$ so that $\{\phi_{jk}(t)\}_{k\in\mathbf{Z}}$ is an orthonormal basis of the wavelet subspace $\mathcal{P}_j$. The mother wavelet function corresponding to $\phi(t)$ is denoted by $\psi(t)$. Then, $\{\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k),\ j,k \in \mathbf{Z}\}$ forms an orthonormal basis in $L^2(\mathbf{R})$. For any $f(t) \in L^2(\mathbf{R})$, we have

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} b_{j,k}\psi_{jk}(t). \qquad (1)$$

The projection $f_J(t)$ of $f(t)$ in $\mathcal{P}_J$ can be written as

$$f_J(t) = \sum_{k=-\infty}^{\infty} c_{J,k}\phi_{Jk}(t) = \sum_{j<J} \sum_{k=-\infty}^{\infty} b_{j,k}\psi_{jk}(t). \qquad (2)$$

We call $f_J(t)$ a scale-limited signal, since its wavelet coefficients are zero for $j \geq J$.

We adopt the following norm notation

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt,$$

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and

$$\|f(t)\|_T^2 = \int_{-T}^{T} |f(t)|^2 dt.$$

The problem of extrapolating noisy data can be stated as the recovery of a scale-limited signal $f(t) \in \mathcal{P}_J$ based on observed noisy data

$$g(t) = f(t) + n(t), \quad t \in [-T, T].$$

where $n(t)$ is zero-mean white noise with energy

$$\|n(t)\|_T^2 \leq \epsilon^2.$$

We consider a constrained minimum norm solution in $\mathcal{P}_J$, i.e.

$$f^* = \min_{f(t)\in\mathcal{P}_J} \|f(t)\|^2, \quad \text{and} \quad \|f^* - g\|_T^2 \leq \epsilon^2. \qquad (3)$$

By using the Lagrangian multiplier, this is equivalent to the minimization problem:

$$\min_{f\in\mathcal{P}_J} \left\{ \|f\|^2 + \mu \left(\|f - g\|_T^2 - \epsilon^2 \right) \right\}, \qquad (4)$$

where μ is the regularization parameter. By using the definition in [10, page 51], $\|f\|^2$ is a stabilizing functional and we call the solution of (4) a "regularized solution".

We present some basic results to characterize the regularized solution below. For more details, we refer to [4] and [5]. In analogy with the band-limited time-concentrated operator, we define the scale-limited time-concentrated operator H as an integral operator which maps $f(t) \in L^2(\mathbf{R})$ to $g(t) \in L^2[-T, T]$ via

$$\sum_{k=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(s)\phi_{Jk}(s)\, ds \right) \phi_{Jk}(t) = g(t), \qquad (5)$$

where $t \in [-T, T]$ and $\phi(t)$ is the scaling function. In words, this operator projects the function $f(t)$ into the wavelet subspace $\mathcal{P}_J$ and then truncates the projected function in the time domain.

Theorem 1 *Let $r_k(t), k \geq 0$, be the eigenfunctions of the scale-limited time-concentrated operator with eigenvalue λ_k and*

$$g(t) = \sum_{k\geq 0} e_k r_k(t) \qquad \text{for } t \in [-T, T].$$

Then, the regularized solution to the noisy extrapolation problem is of the form

$$f^*(t) = \sum_{k\in\mathbf{K}} \frac{\lambda_k \mu e_k}{1 + \lambda_k \mu} \hat{r}_k(t), \qquad (6)$$

where the regularization parameter μ is the solution of

$$\sum_{k\in\mathbf{K}} \frac{\lambda_k e_k^2}{(1 + \lambda_k \mu)^2} = \epsilon^2. \qquad (7)$$

Proof: See [5]

3. REGULARIZED ITERATIVE EXTRAPOLATION ALGORITHM

The regalirized form as given in (6) in not practical in numerical implementation due to the expensive cost in computing the eigenfunctions and their corresponding eigenvalues. It is therefore important to seek an iterative method to compute the regularized solution numerically. Based on the regularization theory [9], we can adopt the following iterative process:

$$\text{Initialization} \quad f_0(t) = 0. \qquad (8)$$

For $n = 0, 1, 2, \cdots$,

$$h_{n+1}(t) = (1 - \alpha) f_n + \alpha\mu(g - \mathbf{P}_T f_n), \qquad (9)$$

$$f_{n+1}(t) = \int_{-\infty}^{\infty} h_{n+1}(s) Q_J(s, t) ds. \qquad (10)$$

By using the eigenfunctions of the scale-limited time-concentrated operator to analyze the above iterative procedure, we obtain the following theorem.

Theorem 2 *Assume that μ is given and*

$$0 < \alpha < \frac{2}{1 + \mu\lambda_k}.$$

The iteration (8)-(10) converges to the regularized solution in Theorem 2. That is, $\lim_{n\to 0} f_n(t) = f^*(t)$.

Proof: See [5].

One implementational issue is the computation of the regularization parameter μ required in (9). Note that the choice of regularization parameter μ depends on the noise energy. When noise is small, we need a large μ value so that the regularized solution will be close to the observed signal. On the other hand, for a larger noise level, we need a smoothness constraint on the solution to stabilize the ill-posed problem. This however reduces the accuracy of the regularized solution. Thus, the choice of an appropriate parameter plays an important role in a regularization procedure.

To compute μ with (7) is expensive since the eigenvalues of the scale-limited time-concentrated operator are needed. Thus, we seek an approximating regularization parameter close to the one given by (7). We can express the regularized solution $f(t)$ in (6) explicitly as a function of μ and t and examine

$$N(\mu) = \|f(\mu, t)\|^2 = \sum_{k\in\mathbf{K}} \frac{(\lambda_k \mu e_k)^2}{(1 + \lambda_k \mu)^2},$$

$$E(\mu) = \|f(\mu, t) - g(t)\|_T^2 = \sum_{k\in\mathbf{K}} \frac{\lambda_k e_k^2}{(1 + \lambda_k \mu)^2}.$$

Note that $N(\mu)$ and $E(\mu)$ are monotonically increasing and decreasing functions of μ, respectively. The continuous curve consisting of $(N(\mu), E(\mu))$, $\mu \geq 0$, is called the L-curve. The importance of this curve was first discussed by Miller [6]. Recently, Hansen and O'Leary [3] used the L-curve to determine the regularization parameter. It was shown in [3] that the L-curve is concave and there exists a sharp corner on this curve which gives the optimal regularization parameter.

Our idea to estimate μ is to first determine its upper and lower bounds and to obtain an initial estimation based on these bounds. Then, we exploit the monotonicity and concavity of the L-curve and apply a linear search method. With this approach, a good approximation of μ can be obtained by only a few iterations.

4. NUMERICAL EXPERIMENTS

Numerical examples are given below to illustrate the performance of the proposed regularized extrapolation algorithm. We use the orthogonal and compactly supported coiflet of order $N = 10$ [2] as the wavelet basis for signal modeling. The coiflet mother wavelet is nearly symmetric around the y-axis so that the filter bank implementation consists of almost linear-phase filters. The high order of vanishing moments implies the smoothness of the waveform and the compact support property makes the implementation easy. Consider a scale-limited sequence $x[n]$ generated by randomly choosing the wavelet coefficients $c_{J,k}$ with $J = 1$ and $-3 \leq k \leq 4$ while setting other wavelet coefficients to zero for the coiflet basis functions. A synthesized clean signal observed at the scale $J_s = 4$ is plotted in Fig. 1. Then, the signal is corrupted by zero-mean additive white Gaussian noise with $SNR = 8$ and we assume that 81 (i.e. $M = 40$) observed noisy data points are available as given in Fig. 1. In this experiment, we avoid the signal modeling problem by assuming that the scale-limited information is partially known a prior, i.e. only $c_{J,k}$ with $J = 1$ and $-3 \leq k \leq 4$ are nonzeros and the wavelet basis is coiflet. However, the exact values of these coefficients $c_{J,k}$ are not known. The extrapolated results by using the regularization approach with the regularization parameter $\mu = 10$, 50 and 10^5 are shown in Figs. 2-4. It is clear that for $\mu = 10$ we have a oversmoothed result. In contrast, the solution for $\mu = 10^5$ is divergent and the case with $\mu = 50$ gives the best result.

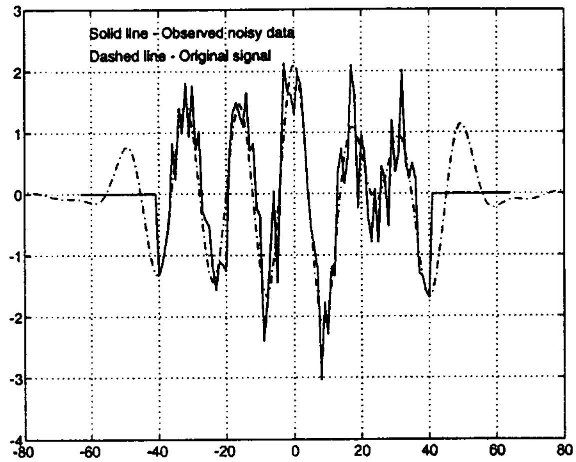

Figure 1: Test Problem: the original signal and observed noisy data with $M = 40$ and $SNR = 8dB$.

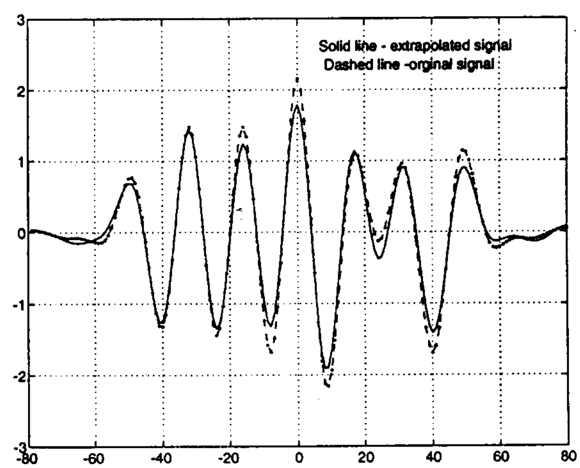

Figure 2: Extrapolated signal with regularization for $\mu = 10$ in 200 iterations.

5. CONCLUSION

Compared with the regularized band-limited extrapolation, the major advantage of this new extrapolation

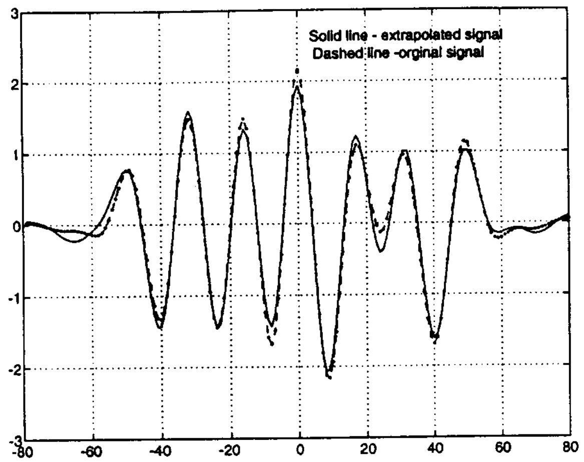

Figure 3: Extrapolated signal with regularization for $\mu = 50$ in 200 iterations.

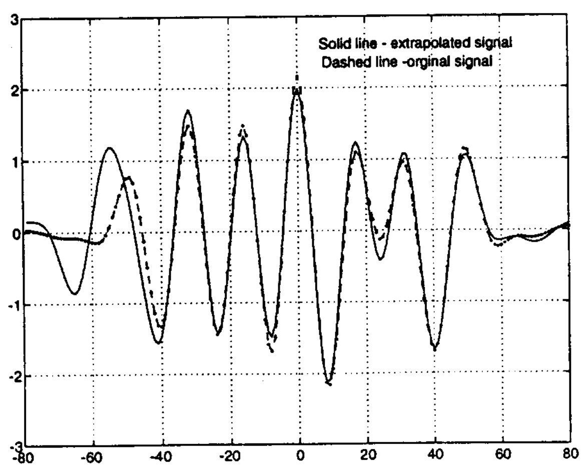

Figure 4: Extrapolated signal with regularization for $\mu = 10^5$ in 200 iterations.

approach is that it provides a large class of wavelet bases for signal modeling while the regularization technique can still be easily incorporated to overcome the ill-posedness of the extrapolation process. Since the wavelet representation is time-frequency localized, the model can be very flexible by adjusting the time and scale parameters. It is particularly suitable for modeling nonstationary signals.

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