SUCCESSIVE PROJECTIONS-LIKE ALGORITHMS FOR SIGNAL APPROXIMATION/ZERO-ERROR MODELLING

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ABSTRACT

In this paper, we show how successive projection-like algorithms may be used for approximation or exact modelling of a signal. For that purpose, we propose a new efficient algorithm providing adequate linear difference equations satisfied by the original signal. The projection operators at each step of the approximation algorithm and the new procedure are shown to be orthogonal. Finally, a pyramidal structure summarises the possibilities offered by the combinations of both procedures.

1. Introduction

In this paper, we are interested in approximating sequences (discrete signals) by a finite sum of complex exponentials on a compact interval. By exponentials, we denote functions of the form $x[n] = \alpha\lambda^n$, with $(\alpha, \lambda) \in \mathbb{R}^2$ or $\mathbb{C}^2$. The probabilistic approach of this problem is classical in signal processing, in the case where *a priori* statistics on the noise are given. The deterministic point of view presents different difficulties related to non linear optimisation. In recent papers [1], [2], [4], [6], [7]. New methods were presented to solve the problem of approximation by sums of a given number of complex exponentials. The problem was formulated in terms of matrix approximation using the Frobenius norm. Then, a method of successive projections was used and provided satisfactory results (see [1], [7] and [6]). It was also shown in [5] that a *zero error* modeling could be obtained for a certain number of involved exponential functions. General results about successive projections algorithms may be found in [3] and references within.

The aim of this communication is to present a general framework for signal approximation and asymptotic *zero error* modelling by the means of successive projections algorithms. First the approximation problem is presented. Then we introduce a new algorithm for finding an asymptotically exact model of the original signal. Finally, a general framework is given for signal approximation and modelling, using the structure of the classical successive projection algorithm.

2. The approximation problem

2.1. Preliminary remarks

Let us consider a finite sequence of data, $s[n]$, $n \in \mathbb{N}$. Suppose that s verifies the following relation:

$$a_p s[n-p] + a_{p-1}s[n-p+1] + \cdots + a_0 s[n] = 0 \ ,\forall n \in \mathbb{Z} \quad (1)$$

with $a_p \neq 0$ and $a_0 \neq 0$.

Let λ_i, $i \in [1..I]$ be the roots of the polynomial $p(z) = a_0 z^p + a_1 z^{p-1} + \cdots + a_p$ with multiplicity α_i. Then the theory of finite difference equations states that s is a linear combination of functions $n^{\alpha_1 - 1}\lambda_1^n$, $\cdots$, λ_1^n, $\cdots$, $n^{\alpha_I - 1}\lambda_I^n$, $\cdots$, λ_I^n.

Let us define the compactly supported sequence $a[n]$, $n \in \mathbb{N}$, by:

$$\begin{array}{lll} a[n] = a_n & \text{if} \quad n \in [0..p] \ , \\ a[n] = 0 & \text{otherwise} \ . \end{array} \quad (2)$$

Then Equation (1) is equivalent to

$$s * a = 0 \ , \quad (3)$$

where '0' denotes the null sequence.

On a compact interval $[a..b]$, equations (1) and (3) are equivalent to the matrix equation:

$$\boldsymbol{a}^t \boldsymbol{S} = \boldsymbol{0} \ , \quad (4)$$

with

$$\boldsymbol{S} = \begin{bmatrix} s[a] & s[a+1] & \cdots & s[b-p] \\ s[a+1] & \cdots & \cdots & s[b-p+1] \\ \vdots & & & \vdots \\ s[a+p] & \cdots & \cdots & s[b] \end{bmatrix} ,$$

and

$$a^t = [a_p, \cdots, a_0] \ .$$

2.2. Reduced rank approximation

As can be seen, the signal s can be written as a sum of exponential or polynomial exponential functions of the form $n^\alpha \lambda^n$ iff the matrix $\boldsymbol{S}$ is not full rank. Of course, in the general case, this is not verified, and the matrix $\boldsymbol{S}$ must be approximated by a matrix with reduced rank. A successive projection algorithm has been proposed in [1], and further discussed in [2],[6], [7], which provides an approximating matrix $\hat{\boldsymbol{S}}$, with neglectable lower singular values. We may now briefly describe the procedure for a better understanding of the new proposed algorithms. The computation of the SVD of $\boldsymbol{S}$ allows an orthogonal decomposition as follows:

$$\boldsymbol{S} = \sum_{i=1}^{p+1} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^t \tag{5}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{p+1} \geq 0$. Then,

$$\boldsymbol{S} = \boldsymbol{E}_1 + \boldsymbol{E}_2 \ , \tag{6}$$

with

$$\boldsymbol{E}_1 = \sum_{i=1}^{h} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^t$$

and

$$\boldsymbol{E}_2 = \sum_{i=h+1}^{p+1} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^t \ .$$

It is well known that $\boldsymbol{E}_1$ is the projection of $\boldsymbol{S}$ on the set of matrices of rank less than or equal to h, denoted *Red*. An iteration of the classical successive projections algorithm is complete when $\boldsymbol{E}_1$ is projected on the space of Hankel matrices, denoted *Hankel*. Finally, we have the following procedure:

$$\mathrm{S}^{(k+1)} = proj_{Hankel}(proj_{Red}(\mathrm{S}^{(k)})) \ , \qquad \mathrm{S}^{(0)} = \mathrm{S} \ . \tag{7}$$

2.3. Equivalence with a null phase filter bank

Let $s^{(k)}$ be the signal associated to the matrix $S^{(k)}$, $u_j^{(k)}$ the compactly supported sequence constructed like $a[n]$ in Section 2.1, and $u_j^{(k)*}(n) = u_j^{(k)}(p+2-n)$. It is shown in [4] that one iteration of the classical algorithm is equivalent to the convolution Equation (8):

$$s^{(k+1)} = (\sum_{j\in[1,r]} u_j^{(k)} * u_j^{(k)*}) * s^{(k)} \ . \tag{8}$$

This remark will be of great importance in the rest of the paper.

3. A new algorithm for zero error modelling

In this section, we propose a new algorithm for asymptotically exact modelling. We first introduce the linear projection operators used in the classical and the novel procedure. Then, we present the complete algorithm for zero error modelling and we study its convergence properties.

3.1. Two well known orthogonal projection operators

Let us define

$$\boldsymbol{U}_1 = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_h] \ ,$$

and

$$\boldsymbol{U}_2 = [\boldsymbol{u}_{h+1}, \ldots, \boldsymbol{u}_{p+1}] \ ,$$

and their associated projection operators, defined for any matrix $\boldsymbol{M}$ with adequate dimensions:

$$\boldsymbol{p}_1(\boldsymbol{M}) = \boldsymbol{U}_1 \boldsymbol{U}_1^t \boldsymbol{M} \ ,$$

and

$$\boldsymbol{p}_2(\boldsymbol{M}) = \boldsymbol{U}_2 \boldsymbol{U}_2^t \boldsymbol{M} \ .$$

Since $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_h$ are the eigenvectors related to the higher singular values, $\boldsymbol{p}_1$ may be considered as an approximation operator, and obviously $\boldsymbol{p}_2$ provides the error of the approximation. It is straightforward to see that $range(\boldsymbol{p}_1)$ and $range(\boldsymbol{p}_2)$ are two orthogonal spaces. Using this notations, indexed by the iteration number k, the classical algorithm can be rewritten:

$$\boldsymbol{S}^{(k+1)} = proj_{Hankel}(\boldsymbol{p}_1^{(k)}(\boldsymbol{S}^{(k)})) \ , \qquad \boldsymbol{S}^{(0)} = \boldsymbol{S} \ . \tag{9}$$

3.2. The new algorithm

The new algorithm aims to find a compactly supported sequence $a[n]$, $n \in \mathbb{N}$ which vanishes when convolved with $s[n]$, $n \in \mathbb{N}$. For that purpose, instead of using the approximation operator $\boldsymbol{p}_1^{(k)}$, at iteration (k), we use the orthogonal operator $\boldsymbol{p}_2^{(k)}$. In this case, we have:

$$\boldsymbol{Z}^{(k+1)} = proj_{Hankel}(\boldsymbol{p}_2^{(k)}(\boldsymbol{Z}^{(k)})) \ , \qquad \boldsymbol{Z}^{(0)} = \boldsymbol{S} \ . \tag{10}$$

We then obtain the following proposition:

Proposition 1 *the algorithm, as defined above, converges to the null matrix and the Frobenius norm of the matrices $\boldsymbol{Z}^{(k)}$ decreases exponentially.*

Proof:

Let $\|.\|$ denote the Frobenius norm of matrices. It is well known to write

$$\left\| \boldsymbol{Z}^{(k)} \right\|^2 = \sum_{i\in[1,p+1]} \sigma_i^2(k) \ ,$$

where the $\sigma_i^2(k)$ are indexed in the decreasing order. Now we note that

$$\left\|p_2^{(k)}(Z^{(k)})\right\|^2 = \sum_{i\in[h+1,p+1]} \sigma_i^2(k) \ ,$$

which implies that

$$\left\|p_2^{(k)}(Z^{(k)})\right\|^2 \leq \frac{p+1-h}{p+1} \left\|Z^{(k)}\right\|^2 \ .$$

Knowing that a consequence of the projection of $p_2^{(k)}(Z^{(k)})$ on the space of Hankel matrices is that

$$\left\|Z^{(k+1)}\right\| \leq \left\|p_2^{(k)}(Z^{(k)})\right\| \ ,$$

we obtain

$$\left\|Z^{(k+1)}\right\|^2 \leq \frac{p+1-h}{p+1} \left\|Z^{(k)}\right\|^2 \ .$$

This also equivalent to

$$\left\|Z^{(k)}\right\| \leq \left(\frac{p+1-h}{p+1}\right)^{\frac{k}{2}} \left\|Z^{(0)}\right\|$$

which ends the proof . □

Due to the equivalence of such algorithms with filter bank processing, we obtain after K iterations:

$$z^{(K)} = f^{(K)} * z^{(0)} \ , \tag{11}$$

where
$f^{(K)}[n]$, $n \in I\!N$ denotes

$$f^{(K)} = \left(\sum_{j\in[h+1,\ldots,p+1]} u_j^{(K-1)} * u_j^{(K-1)*}\right) * \cdots \\ \cdots * \left(\sum_{j\in[h+1,\ldots,p+1]} u_j^{(0)} * u_j^{(0)*}\right) \ . \tag{12}$$

Then, Proposition 1 implies that, after a sufficiently high but finite number of iterations, the original signal nearly belongs to the kernel of a discrete convolution operator involving a signal with compact support. We can consequently approximate the original signal by a linear combination of exponential functions.

As shown before, the exponential functions are the zeros of $F^{(k)}(z)$, the $z-$transform of $f^{(k)}$. The coefficients of these exponentials may be computed using a least square method.

4. A pyramid of approximation/zero error modelling

The existing projection algorithms combined with the one described in this paper offer multiple possibilities for signal modelling:

1. successive approximations of the original signal by considering the largest singular values of the Hankel matrix associated to the signal,
2. construction of filters with compact support for zero error modelling by considering the lowest singular values.

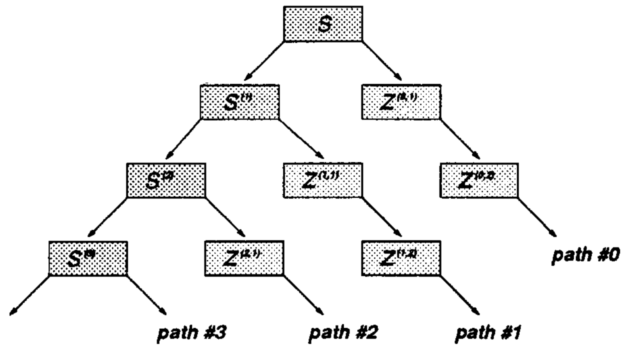

Figure 1: pyramid of approximation/zero-error modelling

The possibilities proposed are represented in the pyramid of figure 1: ↙ represents an iteration using the projection operator $p_1^{(k)}$ and ↘ represents an iteration using the projection operator $p_2^{(k)}$. In fact, path number k involves k iterations of the signal approximation algorithm whereas the number of iterations of the zero error modelling procedure is the same for all paths.

This pyramid shows that successive projection algorithms may be used for both signal approximation and/or zero-error modelling.

5. Simulation results

The algorithm for zero-error modelling is applied to the signal in Figure 2, following path number 1. The SNR is given in Figure 5, as a function of the iteration number, in the case where all the exponential functions are kept. Then, we show the SNR after 10 iterations of the exact modelling procedure, for several pathways. Thanks to S. Mallat's matching pursuit algorithm [8] only 100 functions are selected for reconstruction. The results are given in Figure 4. It has been observed that results are improved if the approximation algorithm is applied before the exact modelling procedure. The best reconstructed signal was obtained using path 7 with only 100 functions kept, as shown in Figure 3.

6. References

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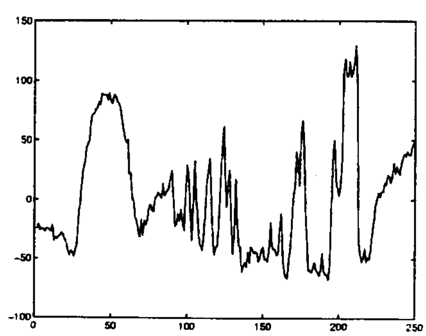

Figure 2: original signal.

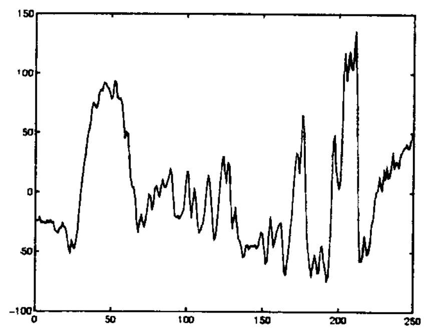

Figure 3: best reconstructed signal (path number 7, 100 functions kept).

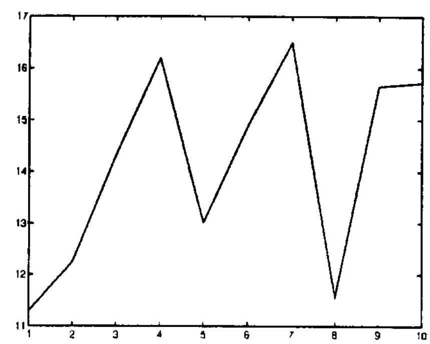

Figure 4: SNR(dB) vs path in the pyramid (100 functions kept).

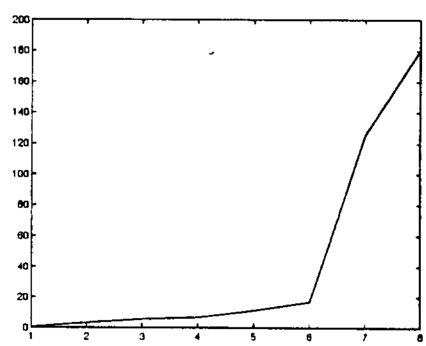

Figure 5: SNR(dB) vs iteration number using path number 1 (all functions kept).