

NONLINEAR RECOVERY OF SPARSE SIGNALS FROM NARROWBAND DATA

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ABSTRACT

This paper describes the connection between a certain signal recovery problem and the decoding of Reed-Solomon codes. It is shown that any algorithm for decoding Reed-Solomon codes (over finite fields) can be used to recover wide-band signals (over the real/complex field) from narrow-band information.

1. INTRODUCTION

In several branches of applied science often observations are bandlimited, even though the underlying phenomena being studied are wideband. Recently it was shown that an uncertainty principle is at the heart of several algorithms [3] for wideband signal recovery from narrowband data [2]. The uncertainty principle in question states that if a finite sequence of length N has N_t non-zero "time" samples and N_f non-zero "frequency" samples (i.e., samples of its N -point Discrete Fourier Transform), then $N_t N_f \geq N$. This result can be used to show that if one loses at most N_f frequency samples of a signal with N_t non-zero samples and $N_t N_f < 2N$, then the signal can be recovered from the remaining frequency samples. Our paper shows that a signal with at most N_t frequency samples can be recovered from any contiguous band of $2N_t$ frequency samples. We show a relationship between this signal recovery question and the decoding of Bose-Chaudhury-Hocquenghem (BCH) codes over finite fields. This relationship also gives an efficient non-linear algorithm for this type of signal recovery.

2. MAIN RESULT

The author was first exposed to the discrete-time uncertainty principle in a paper by Donoho and Stark, where an elementary proof is also given [2]. The result depends on two key ideas; an *interpolation property* and the *pigeonhole principle*.

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Both the DFT and the inverse DFT (of a signal over \mathbb{R} or \mathbb{C} or for that matter any finite field $GF(q)$) can be interpreted as polynomial evaluation at the roots of unity. Hence from the unique interpolation property of polynomials one can infer that frequency samples over any contiguous band of n frequencies can be chosen to uniquely interpolate a signal on any subset of n time sample points. Therefore, if a signal has N_t non-zero values, then necessarily there can be no band of N_t zeros in frequency.

Let G_t and G_f denote respectively the maximal gap (or run of zeros) in time and frequency respectively for a signal.

Lemma 1 (Interpolation Property) $N_t \geq G_f + 1$ and $N_f \geq G_t + 1$

This result can be interpreted for the finite field discrete Fourier transform, where, it the result is commonly referred to as the Bose-Chaudhury-Hocquenghem (BCH) bound [1]: *a vector over the finite field $GF(q)$ with G_f consecutive zeros in spectrum has a Hamming weight greater than G_f .*

Let G_t denote the gap in time for a signal vector of length N . There exists at least one non-zero element in any set of $G_t + 1$ time samples. By dividing up the signal into contiguous blocks of size $G_t + 1$, we infer that $N_t \geq N/(G_t + 1)$.

Lemma 2 (Pigeonhole Principle) $N_t(G_t + 1) \geq N$ and $N_f(G_f + 1) \geq N$.

Combining the two results above gives the uncertainty principle: for any non-zero signal

$$N \leq N_t(G_t + 1) \leq N_t N_f \quad (1)$$

The uncertainty principle is tight whenever N_t (or equivalently N_f) divides N . Indeed, if we choose a Dirac comb as the signal, its DFT is also a Dirac comb with $N_f = N/N_t$ non-zero frequency samples. The uncertainty principle gives lower bounds on N_t and N_f respectively as a function of the other and N and an upper bound on N as a function of N_f and N_t . The

only signal that violates the constraint is the trivial zero signal. This property can be used to show the uniqueness of certain signal recovery problems.

Consider the DFT pair (x, \hat{x}) with N_t being the number of non-zero elements of x . Let the observed data y be bandlimited to band $B \subset \{0, \dots, N-1\}$; that is $y = P_B x$, where P_B is the bandlimiting projection operator. If $|B^c| < N_f$ and $N_t N_f < 2N$, then x can be uniquely recovered from y . For this, it suffices to show that the operator P_B is one-to-one for signals with time-support at most N_t . Indeed if x' is any other signal such that $y = P_B x'$, then $P_B(x - x') = 0$ and $x - x'$ would be a signal with support at most $2N_t$ with at most N_f non-zero frequency samples. By the uncertainty principle, $x - x' = 0$, and hence the result. In [2], a non-linear algorithm for this recovery is given. Moreover, the algorithm is shown to be robust in the presence of noise.

The result above suggests that a signal can be recovered provided one does not lose too much information. We now prove a result, similar in spirit, which says that signals can be recovered provided one retains enough information. More precisely, a signal x with N_t non-zero samples can be recovered from $P_B x$ for any contiguous band B of $2N_t$ or more points. The key assumption required for our result is the contiguity of the band; which is usually not a serious practical limitation. The basic idea is to show that P_B is one-to-one for such signals. Indeed we have (for any other signal x' that satisfies the data) $P_B(x - x') = 0$. Since B is a contiguous band, from the interpolation property (Lemma 1) $x - x'$ has to have $2N_t + 1$ non-zero samples or be zero. Since $x - x'$ can have at most $2N_t$ non-zero samples, necessarily $x = x'$. The result can be interpreted as follows: Given a signal with N_t non-zero samples, one needs to know N_t locations and N_t values (for a total of $2N_t$ numbers). The $2N_t$ frequency samples from any contiguous band can be used to recover this information.

Theorem 1 *A signal with N_t non-zero values can be recovered from any contiguous band of $2N_t$ frequency values.*

We now contrast the Donoho-Stark result with our result. Consider a signal vector of length $N = 1000$ that is 95% sparse; that is, $N_t/N = .05$. Then, the Donoho-Stark result says that if we lose less than $2N/N_t = 40$ samples we can recover the signal. In other words we require at least 961 (not-necessarily contiguous) frequency samples. However, our result above states that we need exactly $2N_t = 100$ frequency samples in any contiguous band. This advantage comes at a price. The non-linear recovery problem that we propose is not robust to noise. All algorithms that we

know of to recover the signal involve polynomial root-finding in some disguised form, with roots on a sparse subset of the unit circle; and this problem can potentially be numerically ill-conditioned.

3. DECODING ALGORITHM

There is a close relationship between this signal recovery problem and BCH coding in information theory; for simplicity we take the example of a Reed-Solomon code that closely fits our results. A Reed-Solomon code is a linear subspace of $GF^{q-1}(q)$ whose Discrete Fourier Transform (over $GF(q)$), which takes on values in $GF(q)$, is zero in a contiguous band of $2N_t$ frequencies. Each vector in this space is a codeword. The code is a $q - 1 - 2N_t$ dimensional subspace of $GF^{q-1}(q)$. Let a codeword c be corrupted by additive noise $e \in GF^{q-1}(q)$ to produce $r \in GF^{q-1}(q)$: $r = c + e$. If the Hamming weight of e (the number of non-zero entries in e) is at most N_t , then the BCH bound implies that c can be recovered from r . An equivalent way of seeing this is that if e has at most N_t non-zero values, then, e (equivalently c) can be recovered from r .

Our signal recovery problem is precisely the Reed-Solomon decoding problem case in the complex field, \mathbb{C} , instead of the finite field, $GF(q)$. One readily verifies that the nature of the field is irrelevant to the problem. Therefore any decoding algorithm for the Reed-Solomon code also gives an algorithm for the signal recovery problem above. There are several algorithms for decoding Reed-Solomon codes in the literature, each giving a constructive solution to our signal recovery problem. One of the more efficient algorithms for this signal recovery problem is the Berlekamp-Massey-Forney algorithm [1]. We briefly comment on another algorithm, the Peterson-Gorenstein-Zierler (PGZ) algorithm, which, while computationally more expensive, has a simple and instructive structure. It solves the problem in two steps: a) compute the locations of the N_t non-zero samples of the signal and b) compute the values at those locations. If we knew the locations of the N_t non-zero samples, it is clear that the values can be obtained by solving a set of linear equations. The time index set $\{0, 1, \dots, N-1\}$ can be identified with the N^{th} roots of unity so that the N_t locations correspond to a subset of the N^{th} roots of unity. The PGZ decoder constructs a polynomial from the given data ($2N_t$ frequency samples) whose roots are precisely this subset of N_t roots of unity. If N is very large the algorithm is not very useful because, then, even in the absence of noise, numerical errors in polynomial root-finding could result in a "wrong" set of N_t locations. These problems occur only for moderate values of N_t .

since the degree of the polynomial is $N_t - 1$. Also since the algorithm is based on solving for $2N_t$ unknowns with $2N_t$ non-linear equations it cannot be robust to noise in the data.

We give the matlab code that implements the Peterson-Gorenstein-Zierler decoder. For the interested reader a matlab mex file (written in C) and implementing the Berlekamp-Massey-Forney algorithm can be obtained from the author.

```
function [x,locations,values]=pgzdecode(X,N,k);
%function [x,locations,values]=pgzdecode(X,N,k);
%Input: X: Consecutive frequency values
%       from frequency k to k+length(X)-1
%       N: Frequency grid spacing is 2*pi/N
%       k: start frequency
%Output:
%       x: Sparse signal.
%       locations: Support locations of x
%       values: Non-zero values of x
%File Name: pgzdecode.m
%Last Modification Date: 4/13/94 09:41:33
%File Creation Date: Sat Feb 26 11:59:55 1994
%Author: Ramesh Gopinath
if nargin<3, k=0;end;
if rem(length(X),2)==1,
    error('Vector length must be even');
end;
n = numerr(X);n2 = 2*n;
M = hankel(X(1:n),X(n:n2-1));
XX = -X(n+1:n2);
Lambda = flipud([M\XX;1]);
locations=round(N*(mod(imag(-log(roots(Lambda)))/(2*pi),1))');
locations = mod(locations,N);
k = (k:k+n-1)';
W = exp(-(sqrt(-1)*2*(pi/N)*k)*locations);
values = (W\X(1:n));
if norm(imag(values)<1e-8) values = real(values);end;
x(locations+1)=values;
end;
```

```
function n = numvals(s);
%function n = numvals(s);
%Input:
%       s: Band of frequency values.
%Output:
%       n: # of non-zero values in time-domain.
%File Name: numvals.m
%Last Modification Date: 3/15/94 13:53:57
%File Creation Date: Sat Feb 26 11:55:54 1994
%Author: Ramesh Gopinath
n2 = length(s);
n = n2/2;
while (n > 0)
    if rank(hankel(s(1:n),s(n:n2-1)))==n,
        return;
    else
        n2=n2-2;
        n=n-1;
    end;
end;
```

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[3] F. Santosa and W. W. Symes. Linear inversion of band-limited reflection seismograms. *SIAM J. Sci. Statist. Comput.*, 7:1307-1330, 1986.