RLS DESIGN OF POLYPHASE COMPONENTS FOR THE INTERPOLATION OF PERIODICALLY NONUNIFORMLY SAMPLED SIGNALS

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ABSTRACT

The generalized sampling theorem states that any analogue signal whose spectrum is limited to $1/T$ can be exactly recovered from N sequences of samples taken at a rate $2/NT$ and all having a different sampling phase. When $N = 2$, the exact interpolation formula can be derived quite easily. The ideal interpolation filters have infinite impulse responses. *This paper addresses the theoretical question of recovering from the 2 initial sequences, any other sequence taken at the same rate* $1/T$ *and with a different sampling phase.* FIR filters optimized for a mean squared error criterion have been derived [5]. *In the present paper, FIR filters are derived for a least square interpolation error. Moreover, an adaptive implementation is proposed and formulated as a Kalman algorithm. Simulation results obtained for AR processes show the effectiveness of the solution compared to static solutions.*

1. INTRODUCTION

The sampling theorem states that any signal whose spectrum is limited to $f = 1/T$ can theoretically be exactly recovered from a sequence of samples taken at a rate greater than $f_s = 2/T$ [2, 3]. The analogue signal recovery requires filtering the sequence of samples with an infinite length signal. This result is known as the Whittaker interpolation. The sampling theorem can be extended [2, 4]. It can be shown [2] that a signal whose spectrum is limited to $1/T$ can be theoretically exactly recovered from N sequences of samples taken at a rate $2/NT$ and all having different sampling phases. In particular, when $N = 2$, the exact formulas show that as in the Whittaker method, the 2 filters the 2 sequences have to be interpolated with, have infinite duration. It happens [1, 5] that one knows a sequence of samples for some locations or some values of the time (or space), and that one wants to know the signal at other instants or locations. This is a *resampling* problem. In [4], polyphase structures for the reconstruction of a band-limited sequence from a non-uniformly decimated *sequence* are derived. *In this paper, we deal with a one-dimensional situation; it is assumed that we know 2 sequences resulting from sampling with the same rate* $1/T$ *and different phases, a signal whose spectrum is limited to* $1/T$. *Thus, each one of the 2 sequences taken individually suffers from aliasing. Nevertheless, this aliasing can be completely removed by adequately combining the information of the 2 sequences. We deal with the generalized interpolation problem, which is to find from the 2 initial sequences a third one taken at the same rate but with another sampling phase.* These 3 phases can have any value as long as they are different. The 2 ideal filters have infinite length impulse responses. However, in many practical problems, it is required to have finite length impulse response filters. In [5], the problem has been solved for a minimum mean squared error. In the present paper, *we analyze the problem of designing FIR filters which minimize the least square interpolation error [6].* Such a design takes into account the data itself rather than statistical expectations. Besides, adaptive structures can be derived and an extended version of the standard Kalman algorithm will be proposed. Explicit solutions are provided for the 2 sets of coefficients.

2. PROBLEM DEFINITION

We denote by $x_a(t)$ the analogue signal, by $x_s(t)$, its sampled version, and by $x(n)$, the corresponding sequence. Sampling a signal $x_a(t)$ with a period T and a phase T_1 provides a signal given by :

$$x_s(t) = \sum_{k=-\infty}^{\infty} x_a(kT + T_1)\,\delta(t - kT - T_1) \tag{1}$$

The frequency in radian used for analogue signals is denoted by ω and the spectrum of sequences is a periodical function of Ω (period 2π). The signal $x_s(t)$ has a spectrum given by

$$\mathcal{X}_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \exp^{-2\pi jkT_1/T}\ \mathcal{X}_a(\omega - 2\pi k/T) \tag{2}$$

We can establish a relationship between the signals $x_s(t)$ and $x(n)$, by requiring that

$$x(n) = x_s(nT + T_1) \tag{3}$$

The correspondence between the 2 spectra $\mathcal{X}_s(\omega)$ and $\mathcal{X}(\Omega)$ is given by :

$$\mathcal{X}(\Omega) = \exp^{j\Omega T_1/T}\ \mathcal{X}_s(\omega) \tag{4}$$

and $\Omega = \omega T$. We assume to have a signal $x_a(t)$ whose spectrum is limited to $f = 1/T$. We also know 2 sampled versions of this signal, denoted by $x_{1,s}(t)$ and $x_{2,s}(t)$. These signals result from sampling $x_a(t)$ with a rate $1/T$ and respective phases T_1 and T_2. We want to find the filters to be applied on $x_{1,s}(t)$ and $x_{2,s}(t)$ in order to find $x_{3,s}(t)$ which

0-7803-2431-5/95 $4.00

corresponds to the sampling of $x_a(t)$ with the rate $1/T$ and a sampling phase T_3 different from T_1 and T_2. The solution to recover x_a from $x_{1,s}$ and $x_{2,s}$ is given by [7] :

$$\begin{aligned} x_a(t) &= \left[\alpha w^+(t) + \alpha^* w^-(t)\right] \otimes x_{1,s}(t) \\ &- \left[\beta w^+(t) + \beta^* w^-(t)\right] \otimes x_{2,s}(t) \end{aligned} \quad (5)$$

where $\otimes$ stands for convolution, * denotes complex conjugation and α and γ are defined by

$$\alpha = \frac{T \exp^{\pi j(T_1-T_2)/T}}{2j \sin\left[\pi(T_1-T_2)/T\right]} \quad (6)$$

$$\beta = \frac{T \exp^{\pi j(T_2-T_1)/T}}{2j \sin\left[\pi(T_1-T_2)/T\right]} \quad (7)$$

The impulse responses of the 2 filters are given by :

$$w^+(t) = \frac{\exp^{j\pi t/T}}{T} \frac{\sin(\pi t/T)}{(\pi t/T)} \quad (8)$$

$$w^-(t) = \frac{\exp^{-j\pi t/T}}{T} \frac{\sin(\pi t/T)}{(\pi t/T)} \quad (9)$$

These 2 filters $w^+(t)$ and $w^-(t)$ pass only frequencies between $[0, 1/T]$ and $[-1/T, 0]$, respectively. *We can further assume that the 2 interpolation filters come from the modulation of a same lowpass prototype filter.* The 2 filters $w^+(t)$ and $w^-(t)$ are then obtained by shifting on the frequency axis a same transmittance $\mathcal{W}(\omega)$. We then have :

$$\begin{aligned} \mathcal{W}^-(\omega) &= \mathcal{W}(\omega + \omega_0) \\ \mathcal{W}^+(\omega) &= \mathcal{W}(\omega - \omega_0) \end{aligned} \quad (10)$$

with $\omega_0 = \pi/T$. The sequence $x_3(k_3) = x_a(k_3T + T_3)$ is given by :

$$\begin{aligned} x_3(k_3) &= \sum_{k_1=-\infty}^{\infty} x_{1,s}(k_1T + T_1) \\ &\times \frac{\sin\left[\pi(k_3T + T_3 - k_1T - T_2)/T\right]}{\sin\left[\pi(T_1 - T_2)/T\right]} \\ &\times w(k_3T + T_3 - k_1T - T_1) \\ &- \sum_{k_2=-\infty}^{\infty} x_{2,s}(k_2T + T_2) \\ &\times \frac{\sin\left[\pi(k_3T + T_3 - k_2T - T_1)/T\right]}{\sin\left[\pi(T_1 - T_2)/T\right]} \\ &\times w(k_3T + T_3 - k_2T - T_2) \end{aligned} \quad (11)$$

This is the exact interpolation formula, which shows that in the ideal situation, infinite impulse response filters have to be used.

3. LS OPTIMIZED FIR FILTERS

LS designed filters minimize the least squared error between the desired interpolated signal and the approximated version. According to the results of the previous section, one can write that the desired signal has a spectrum given by :

$$\begin{aligned} \mathcal{X}_{3,s}(\omega) &= \mathcal{X}_{1,s}(\omega)\left[\alpha \mathcal{W}^+_{3,1,s}(\omega) + \alpha^* \mathcal{W}^-_{3,1,s}(\omega)\right] \\ &- \mathcal{X}_{2,s}(\omega)\left[\beta \mathcal{W}^+_{3,2,s}(\omega) + \beta^* \mathcal{W}^-_{3,2,s}(\omega)\right] \end{aligned} \quad (12)$$

where the subscript $(3, i, s)$ means sampling with a phase $T_3 - T_i$. Equivalently, we have about the spectra of sequences that :

$$\begin{aligned} \mathcal{X}_3(\Omega) &= (\gamma + \gamma^*)\mathcal{X}_1(\Omega)\mathcal{W}_{3,1}(\Omega - \pi) \\ &- (\delta + \delta^*)\mathcal{X}_2(\Omega)\mathcal{W}_{3,2}(\Omega - \pi) \end{aligned} \quad (13)$$

where $\mathcal{W}_{3,i}(\Omega)$ denotes the spectrum of the sequence produced by sampling the impulse response $w(t) = \text{sinc}(\pi t/T)$ with phase $T_3 - T_i$. The values of γ and δ are given by

$$\gamma = \frac{T \exp^{\pi j(T_3-T_2)/T}}{2j \sin\left[\pi(T_1-T_2)/T\right]} \quad (14)$$

$$\delta = \frac{T \exp^{\pi j(T_3-T_1)/T}}{2j \sin\left[\pi(T_1-T_2)/T\right]} \quad (15)$$

We want to find 2 approximation filters, or polyphase components, denoted by $\hat{w}_{3,1}(n)$ and $\hat{w}_{3,2}(n)$ which minimize the least squared interpolation error. The objective function to be minimized is given by

$$J = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{E}(\Omega)\, \mathcal{E}^*(\Omega) \mathrm{d}\Omega \quad (16)$$

The error spectrum can be written as :

$$\begin{aligned} \mathcal{E}(\Omega) &= a\mathcal{X}_1(\Omega)\left[\mathcal{W}_{3,1}(\Omega - \pi) - \hat{\mathcal{W}}_{3,1}(\Omega - \pi)\right] \\ &- b\mathcal{X}_2(\Omega)\left[\mathcal{W}_{3,2}(\Omega - \pi) - \hat{\mathcal{W}}_{3,2}(\Omega - \pi)\right] \end{aligned} \quad (17)$$

where $\hat{\mathcal{W}}_{3,i}(\Omega)$ is the approximated version $\mathcal{W}_{3,i}(\Omega)$. We also have :

$$\begin{aligned} J &= \begin{bmatrix} \widehat{W}^T_{3,1} & \widehat{W}^T_{3,2} \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} \widehat{W}_{3,1} \\ \widehat{W}_{3,2} \end{bmatrix} \\ &+ \begin{bmatrix} \widehat{W}^T_{3,1} & \widehat{W}^T_{3,2} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + d \end{aligned} \quad (18)$$

where

1. $(a = \gamma + \gamma^*,\ b = \delta + \delta^*)$
2. $\widehat{W}_{3,i}$ is a vector with the ith optimized impulse response,
3. $\Phi_{1,1}$ is a symmetrical matrix whose element (i, j) $(i, j \in [-N, N]$ for impulse responses evaluated in $2N + 1$ points) gives the value in $n = i - j$ of $(-1)^n aa^* \phi_{x_a}(nT)$ and $\phi_{x_a}(nT)$ is the correlation function of the processed signal,
4. $\Phi_{2,2}$ is a symmetrical matrix whose element (i, j) gives the value in $n = i - j$ of $(-1)^n bb^* \phi_{x_a}(nT)$.
5. $\Phi_{1,2}$ is a matrix whose element (i, j) gives the value in $n = i - j$ of $-ab^*(-1)^n \phi_{x_a}(-nT - T_2 + T_1)$.
6. $\Phi_{2,1}$ is a matrix whose element (i, j) gives the value in $n = i - j$ of $-ba^*(-1)^n \phi_{x_a}(-nT - T_1 + T_2)$.
7. B_1 is a matrix whose element $n \in [-N, N]$ is $-2aa^*(-1)^n[(-1)^n w_{3,1}(n) \otimes \phi_{x_a}(nT)] + (a^*b + ab^*)(-1)^n[(-1)^n w_{3,2}(n) \otimes \phi_{x_a}(nT + T_2 - T_1)]$.

8. B_2 is a matrix whose element $n \in [-N, N]$ is $-2bb^*(-1)^n[(-1)^n w_{3,2}(n) \otimes \phi_{x_*}(nT)] + (a^*b + ab^*)(-1)^n[(-1)^n w_{3,1}(n) \otimes \phi_{x_*}(nT + T_1 - T_2)]$.
9. d is a constant computed by evaluating

$$[(-1)^n C_{ww}(n)] \otimes [aa^*\phi_{x_1}(n) + bb^*\phi_{x_2}(n) + ab^*\phi_{x_1x_2}(n) + ba^*\phi_{x_2x_1}(n)]$$

for $n = 0$ where

(a) $C_{ww}(n)$ is the deterministic correlation function of the objective filter $w(n)$ resulting from sampling $w(t)$ with phase T_3,

(b) $\phi_{x_i}(n)$ is the correlation function of x_i,

(c) $\phi_{x_ix_j}(n)$ is the cross-correlation function of x_i and x_j.

By requiring that the derivatives of J with respect to $\widehat{W}_{3,1}$ and $\widehat{W}_{3,2}$ be equal to 0, we find that the optimal solution is given by :

$$\begin{bmatrix} \Phi_{11} + \Phi_{11}^T & \Phi_{12} + \Phi_{21}^T \\ \Phi_{21} + \Phi_{12}^T & \Phi_{22} + \Phi_{22}^T \end{bmatrix} \begin{bmatrix} \widehat{W}_{3,1} \\ \widehat{W}_{3,2} \end{bmatrix} = \begin{bmatrix} -B_1 \\ -B_2 \end{bmatrix} \quad (19)$$

It should be mentioned that the 2 vectors $\widehat{W}_{3,1}$ and $\widehat{W}_{3,2}$ provide the adequately sampled versions of $w(t)$ but not the complete impulse responses. As it can be seen from equation 11, additional factors equal to $\sin[\pi(k_3T + T_3 - kT - T_i)/T] / \sin[\pi(T_1 - T_2)/T]$ have to be taken into account.

4. RECURSIVE ALGORITHM - KALMAN EXTENSION

In the previous section, a static solution has been obtained for the FIR filters. Actually, filters can be derived for a recursive approach as well. In a Kalman approach, the objective function to minimize is

$$J(N) = \sum_{n=1}^{N} e^2(n)\, \lambda^{N-n} \quad (20)$$

where N is the number of data available and $\lambda \leq 1$ is a forgetting factor. The error signal can be written in a vector form

$$e(n) = x_3(n) - \widehat{W}_{3,1}^T(N)X_1(n) - \widehat{W}_{3,2}^T(N)X_2(n) \quad (21)$$

where $x_3(n)$ is the target point, $\widehat{W}_{3,i}^T(N)$ is the optimal impulse response to be obtained from the N data, and

$$X_i(p) = [x_i(p-K), \cdots, x_i(p), \cdots, x_i(p+K)]^T \quad (22)$$

where $2K+1$ is the filter length. By taking the derivatives with respect to the $\widehat{W}_{3,i}^T(N)$ and forcing them to be 0, one obtains a set of linear equations :

$$\begin{bmatrix} \Psi_{11}(N) & \Psi_{12}(N) \\ \Psi_{21}(N) & \Psi_{22}(N) \end{bmatrix} \begin{bmatrix} \widehat{W}_{3,1}(N) \\ \widehat{W}_{3,2}(N) \end{bmatrix} = \begin{bmatrix} B_1(N) \\ B_2(N) \end{bmatrix} \quad (23)$$

or equivalently

$$\Psi(N)\widehat{W}(N) = B(N) \quad (24)$$

where $\Psi_{ij}(N) = \sum_{n=1}^{N} \lambda^{N-n} X_i(n)X_j^T(n)$ and $B_i = \sum_{n=1}^{N} \lambda^{N-n} x_3(n)X_i(n)$. We therefore also have

$$\Psi_{ij}(N) = \lambda\Psi_{ij}(N-1) + X_i(N)X_j^T(N) \quad (25)$$

and we can use the matrix inversion lemma

$$\Psi^{-1}(N) = \frac{1}{\lambda}\left[\Psi^{-1}(N-1) - \frac{\Psi^{-1}(N-1)\,X(N)X^T(N)\,\Psi^{-1}(N-1)}{\lambda + X^T\Psi^{-1}(N-1)X(N)}\right] \quad (26)$$

where $X(N) = \left[X_1^T(N), X_2^T(N)\right]$. A Kalman algorithm can now be defined. It will be omitted for the sake of concision.

The method has been tested as follows. An signal has been produced by means of an AR Markov process with $\sqrt[8]{0.9}$ as correlation. This signal has been lowpass filtered by means of an ideal lowpass filter with 0.235 as cutoff frequency (1 referring to half the sampling rate). An 8-fold downsampling consequently provides a sequence with aliasing as considered in the present work. The algorithm has been tested for $T_1 = 0$, $T_2 = 0.25$ and $T_3 = 0.875$ and for 7 taps for each polyphase component. The forgetting factor has arbitrarily been set to 0.99. The results are as follows. All curves give the absolute value of the relative error versus the sample position. The error is the difference between the interpolated value and the "true" value which can derived by downsampling the output of the process with phase T_3. Figure 1 is obtained for the Kalman algorithm with a windowed version of the ideal filters as a start. Figure 2 is obtained with static filters provided by windowing the ideal interpolation filters and figure 3 is obtained by means of the MMSE designed filters proposed in [5]. The sum of squared error is 0.1411 for the first curve, 1.2981 for the second and 0.2133 for the third one. However, when the number of taps goes down, the difference between the methods associated with figures 1 and 3 decreases slightly.

5. CONCLUSIONS

This paper has been devoted to the problem of interpolation in the context of periodical nonuniform sampling. Ideal interpolation filters are of the infinite impulse response type. In this contribution, finite impulse response filters have been derived for a least square criterion. Moreover an adaptive implementation has been obtained in the form of a Kalman algorithm. Simulations results performed for signals produced by a first-order Markov process show the effectiveness of the method. The adaptive method outperforms the windowed ideal filters or those derived for an MMSE criterion.

Future work will be devoted to how incorporate such a method in a motion compensation scheme for coding interlaced images.

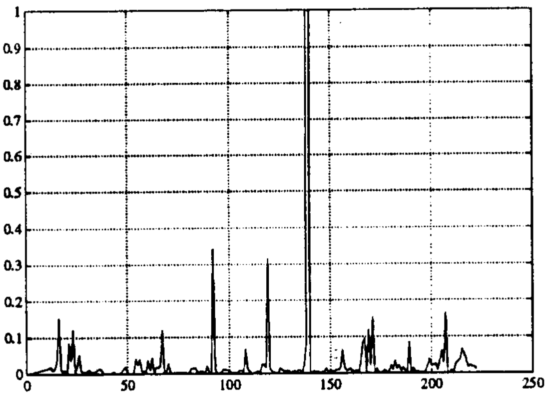

Figure 1. Evolution of the relative error in absolute value for the Kalman algorithm initialized with the values of static filters

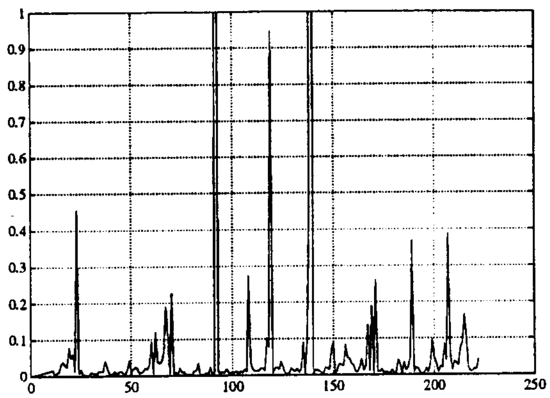

Figure 2. Evolution of the relative error in absolute value for static filters obtained by windowing the ideal filters

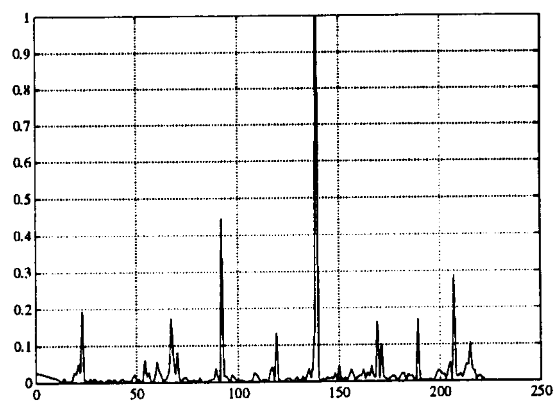

Figure 3. Evolution of the relative error in absolute value for the MMSE designed FIR filters

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