INTERPOLATION OF LOWPASS SIGNALS AT HALF THE NYQUIST RATE

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ABSTRACT

In this letter, we shall describe interpolation of low pass signals from a class of stable sampling sets at half the Nyquist rate. Practical reconstruction algorithms are also suggested.

1. INTRODUCTION

Practical signal recovery from nonuniform samples at rates above the Nyquist are known and are discussed in the literature[1],[2],[3]. Although it was known that nonuniform samples at rates below the Nyquist rate is possible[2], issues such as stability, interpolation, and practical reconstruction techniques have not been addressed. In this letter, we shall discuss a class of stable nonuniform samples at rates close to half the Nyquist rate. We will develop formulas based on Lagrange interpolation and suggest practical methods for recovery.

2. CHOICE OF SAMPLING SET

The analytic signal for a low-pass signal of bandwidth W is defined as

$$x_a(t) = x(t) + j\hat{x}(t) = A(t)\exp[j\theta(t)] \quad (1)$$

$$\tan\theta(t) = \frac{\hat{x}(t)}{x(t)} \quad (2)$$

The nonuniform points are chosen such that

$$x(t_k)\sin\omega_0 t_k = \hat{x}(t_k)\cos\omega_0 t_k\,,\ for\ k \in Z \quad (3)$$

where ω_0 is an arbitrary frequency, which determines the sampling rate. In case the sampling rate is higher than half the Nyquist rate, a subset of samples are chosen such that we get W samples on the average, i.,e., half the Nyquist rate. For example, if $2f_0/w = N$, where N is an integer, then every N^{th} crossings is chosen to satisfy half the Nyquist rate. Eqn(3) is equivalent to the zero-crossings of the single side band modulation of $x(t)$. From (2) and (3), we have

$$\theta(t_k) = \omega_0 t_k + k\pi \qquad k \in Z \quad (4)$$

Deviation of nonuniform points from uniform positions ($t_k - kT$, where T is the sampling interval) decreases if ω_0 is taken to be very large and every $\lceil 2f_0/w \rceil$ (rounded up) sample is chosen. This fact implies that the nonuniform samples are well behaved and can be stable.

If the analytic signal in (1) is shifted to the left by $W/2$ in the frequency domain, we derive the low pass equivalent signal:

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where $\hat{x}$ is the Hilbert transform of $x(t)$, $A(t)$ is the magnitude (envelope) and

$$x_{lp}(t) = x_a(t)e^{-j\pi wt} = x_i(t) + jx_q(t) =$$

$$= A(t)e^{j\phi(t)} \qquad (5)$$

where x_i and x_q are real in-phase and quadrature signals of bandwidth $w/2$, and ϕ(t) from

(1) and (5) is

$$\phi(t) = \theta(t) - \pi wt \qquad (6)$$

From (4) and (6), we derive $\phi(t_k)$ from t_k

$$\phi(t_k) = \omega_0 t_k + k\pi - \pi w t_k \qquad (7)$$

From (5) we have

$$x_i(t_k) = A(t_k)\cos\phi_k =$$

$$= (-1)^k A(t_k)\cos(\omega_0 - \pi w)t_k \qquad (8.1)$$

$$x_q(t_k) = A(t_k)\sin\phi_k =$$

$$(-1)^k A(t_k)\sin(\omega_0 - \pi w)t_k \qquad (8.2)$$

If the sampling rate satisfies half the Nyquist rate, from (8), $x_i(t)$ *and* $x_q(t)$ can be recovered from $A(t_k)$ since their corresponding bandwidths are $w/2$. As a result $x_{lp}(t)$ can be recovered from (5). $x(t)$ can then be recovered from $x_{lp}(t)$, viz.,

$$x(t) = Re[x_{lp}(t)e^{j\pi wt}] \qquad (9)$$

Thus signal recovery from half the Nyquist rate is possible in principle.

3. LAGRANGE INTERPOLATION

The above discussion can be formulated in terms of Lagrange interpolation,i.,e.,

$$x_{lp}(t) = \sum_k x_i(t_k)\psi_k(t) + j\sum_k x_q(t_k)\psi_k(t) \qquad (10)$$

In (10), the sampling rate (r)is assumed to be greater than half the Nyquist rate ($r \geq w$), and $\psi_k(t)$ is the Lagrange interpolation given by

$$\psi_k(t) = \frac{H(t)}{\dot{H}(t_k)(t-t_k)}, \qquad (11)$$

$$H(t) = \prod_k (t - t_k)$$

The convergence of (10) is guaranteed from the discussion after eqn (4) from the fact that a sufficient condition for the convergence of Lagrange interpolation is $|t_k - kT| < T/4$, [2].

From (5),(7),(8) and (10) we have

$$x_{lp}(t) = \sum_k (-1)^k A(t_k)\, e^{jt_k(\omega_0 - \pi w)}\psi_k(t) \qquad (12)$$

From (9) and (12) we have

$$x(t) = \sum_k (-1)^k A(t_k) \cos[\pi w(t-t_k) + \omega_0 t_k]\psi_k(t)$$

(13)

The above formula is the implicit interpolation from nonuniform samples. To derive an explicit interpolation formula, we use (4), (1) and (13) to get an interpolation in terms of

$x(t_k)$ and $\hat{x}(t_k)$.

$$x(t) = \sum_k x(t_k)\cos[\pi w(t-t_k)]\psi_k(t) -$$

$$- \sum_i \hat{x}(t_k) \sin[\pi w(t-t_k)]\psi_k(t)$$

(14)

The implicit formula can then be derived from (2), (4) and (14),i.,e.,

$$\hat{x}(t_k) = x(t_k) \tan(\omega_0 t_k)$$

and

$$x(t) = \sum_k x(t_k)\frac{\cos[\pi w(t-t_k)+\omega_0 t_k]}{\cos[\omega_0 t_k]}\psi_k(t)$$

(15)

In case $\cos(\omega_0 t_k) = 0$, from (3), we know that $x(t_k) = 0$, and

$$x(t_k) / \cos(\omega_0 t_k) = \hat{x}(t_k) / \sin(\omega_0 t_k)$$

The above equation implies that (15) does not blow up and is well behaved.

In practice, we can recover $x_i(t)$ *and* $x_q(t)$ from nonuniform samples using iterative techniques[1],[3] and[4] . $x(t)$ can thus be recovered from (9). Iterative techniques are guaranteed to converge if the sampling set is stable and the sampling set $\{t_k\}$ is operating at rates above w samples/sec (half the Nyquist rate for $x(t)$).

Conclusion

We developed a class of nonuniform samples at rates close to half the Nyquist rate. Iterative techniques can be used to recover a low pass signal from the sampling set.

References

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