OPTIMUM MINIMAX ESTIMATION OF QUADRATIC FUNCTIONALS FOR QUADRATICALLY CONSTRAINED SIGNAL CLASSES

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ABSTRACT

A new procedure for the minimax estimation of quadratic functionals of signals is described. The estimates are optimum when the signals satisfy a quadratic constraint, a common assumption made for estimation of linear functionals. The method will, for example, provide best minimax estimates of signal energy in a time-window and of pointwise evaluations of Fourier transform magnitude, in contrast to earlier methods, which first obtain optimum minimax estimates of *linear* functionals, and subsequently form a suboptimum quadratic estimate by evaluating a weighted sum of the squared linear estimates.

1. INTRODUCTION

Several problems in signal processing involve estimating quadratic functionals of a signal. Examples include the estimation of signal energy in a time-window, estimating the magnitude of the Fourier transform of the signal, or the estimation of semblance for geophysical applications, [1]. The usual procedure is to find optimum estimates of a set of related linear functionals, and subsequently to form an estimate of the quadratic functional by evaluating a weighted combination of the squared linear estimates. Even though optimum estimates of the underlying linear functionals are used, the resulting estimate of the quadratic functional will in general be suboptimal. The procedure described here estimates quadratic functionals optimally, under the hypothesis that the class of signals that are admissible for estimation satisfy a quadratic constraint. The quadratic constraint assumption made here for estimation of quadratic functionals is one that is usually made for minimax linear estimation [2–6], and has some physical basis in geophysical applications.

The example of signal interpolation illustrates the difference between estimating linear functionals and estimating quadratic functionals. The interpolation problem may be viewed as one of estimating a missing sample $x(n)$ of a signal x, given known samples $x(k_i), i = 1, \ldots, M$. The procedure used to estimate $x(n)$ depends on the *a priori* assumptions made. We confine ourselves to a common hypothesis, also made in [2–6], that the unknown signal satisfies a quadratic constraint, described more fully below. The optimum minimax estimation of a single linear functional, such as the missing sample $x(n)$, under such a hypothesis, is a well-known procedure described in [2]. The same procedure applies to joint estimation of a collection of general unknown linear functionals, $z_i, i = 0, \ldots, M$, (or a general linear transformation of the signal): examples of more general linear functionals include the real and imaginary parts of the Fourier transform of x, evaluated at a particular frequency. The question addressed in this paper is the estimation of $|x(n)|^2$, or $\sum_i |z_i|^2$. Such quantities are quadratic functionals, e.g., the sum of squared time-samples in a window, $\sum_{n=0}^{T} |x(n)|^2$. Another example is the magnitude squared of the Fourier transform evaluated at a particular frequency, which is the sum of the squares of the real and imaginary parts. One procedure to estimate $\sum_n |x(n)|^2$ is to compute estimates of each individual $x(n)$ and then to form the sum of the squares of the estimated linear functionals. While each individual estimate of $x(n)$ may be minimax optimum, the estimate of the quadratic sum $\sum |x(n)|^2$ need not be optimum. An alternative procedure, described here, is to seek optimum minimax estimates of the quadratic functional directly.

2. REVIEW OF LINEAR ESTIMATION

Optimum minimax estimation of linear functionals is briefly reviewed: further details may be found in [2–4]. Figure 1 illustrates the abstract setting for the problem,

[3] and uses signal interpolation as an example. The signal x is assumed to belong to an inner-product space X. For the interpolation problem, the signal x might be assumed to be the output of a band-limiting filter H, and the inner-product might be the usual L^2 inner-product. The known measurements are a collection of linear functionals evaluated at x: these form a vector y in a 'data space' Y. They may be viewed as the result of applying an information operator I to x, so that $Ix = y$. In the interpolation problem, the vector y consists of the known samples of the signal. It is desired to estimate a real linear functional of x, denoted Ux, given the known measurements y. The estimate may be viewed as applying an approximation operator A to the known data, to obtain an estimate Ay. The signal interpolation example requires an estimate of a missing sample of x from the known samples of x.

More generally, it may be desired to estimate an arbitrary linear transformation of x, lying in some vector space Z. This may equivalently be viewed as joint estimation of a collection of real linear functionals of x, the coordinates in some basis for Z. For example the estimation of complex linear functionals is equivalent to joint estimation of the real and imaginary parts.

To pose the estimation problem it is necessary to make some additional hypotheses about the signal x. The assumption made here is that x belongs to an ellipsoid in the inner-product space X. This is a quadratic constraint since it is a bound on a weighted sum of squared coordinates, the coordinates being the coefficients that result from expanding the signal in some family of basis functions. Such constraints arise for example in geophysical applications: the constraint may reflect the finite energy that a capacitor supplies to a transducer exciting a sound pulse propagating through a rock formation to an array of recording sensors. For the signal interpolation problem, this constraint arises by assuming that the signal x is the output of a band-limiting filter H, whose inputs have bounded energy, as shown in Figure 1.

The quadratic constraint assumption for the signal class leads to a special geometry, illustrated in Figure 2. The ellipsoidal class, K, comprises the signals that are admissible for estimation. The known measurements of linear functionals, i.e., the *linear* constraints, require that the unknown signal must also belong to the hyperplane $Ix = y$ in the signal space X. Thus the signal x lies in intersection of the hyperplane and the ellipsoid, which we call the hypercircle $C(y)$: as the known measurements y vary, the hyperplane moves around, changing the hypercircle $C(y)$. It can be shown [3, 4] that the hypercircle is of the form

$$C(y) = I^{-1}y + ||1 - I^{-1}y||^2(F \cap K) \qquad (1)$$

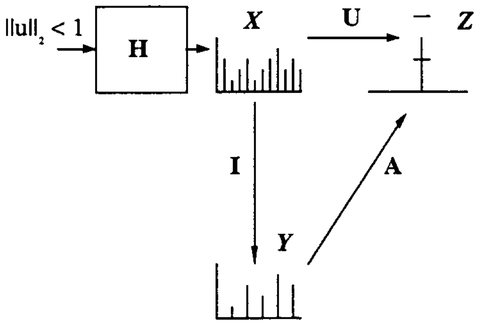

Figure 1: General setting for the estimation problem

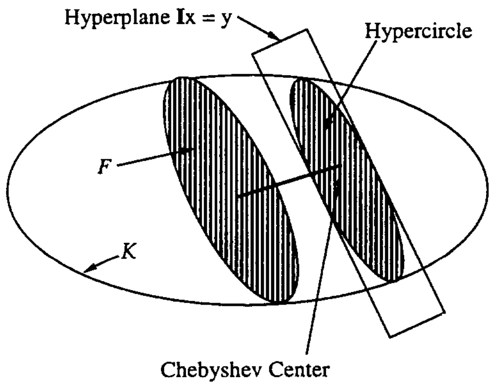

Figure 2: Geometry of the problem

In (1), F is the null space of the information operator I: the subspace of signals x for which $Ix = 0$. Furthermore $I^{-1}y$ denotes the minimum norm signal satisfying $Ix = y$. A key property of $C(y)$ is that it is a translate of $F \cap K$, a convex and balanced set: "balanced" means that if x belongs to $F \cap K$, so does $-x$. It follows that $I^{-1}y$ is the center of the hypercircle in the sense that it minimizes the maximum distance over all points in $C(y)$:

$$\sup_{x \in C(y)} ||I^{-1}y - x|| = \inf_{x' \in X} \sup_{x \in C(y)} ||x' - x|| \qquad (2)$$

As x varies over the hypercircle, a real linear functional of x, Ux (e.g., a time-sample $x(n)$) assumes a range of possible values, illustrated in Figure 3. This range of values is an interval. The optimum minimax estimate for Ux, which is a functional of the data y, denoted Ay, is the center of the interval: it is optimum in the sense that it minimizes the maximum possible error over all values that Ux can possibly take:

$$\sup_{x \in C(y)} |Ay - Ux| = \inf_{z \in Z} \sup_{x \in C(y)} |z - Ux|$$

The geometry of the problem makes the computation of the optimum estimate of Ux simple. The op-

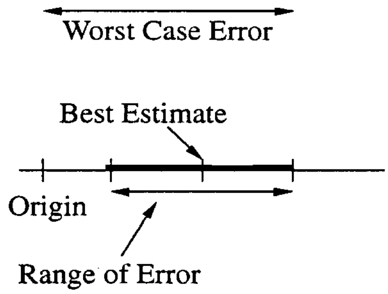

Figure 3: The range of possible values for real linear functional Ux as x ranges of the hypercircle $C(y)$

timum minimax estimate is $U(I^{-1}y)$, a simple consequence of the following observations. Linear transformations map convex balanced sets to convex balanced sets. Applying U, a linear transformation, to (2) shows that $UC(y)$ is also a translate of a convex balanced set, with center $UI^{-1}y$. Thus U maps the center of $C(y)$ to the center of $UC(y)$. In particular, when U is a linear functional, the set of values $U(F \cap K)$ must be an interval on the real line, centered on zero, and $UC(y)$ is a translate of this interval, centered on $UI^{-1}y$.

3. QUADRATIC ESTIMATION

We now consider estimating quadratic functionals of the form $||Ux||_Z^2$, where $||\cdot||_Z$ denotes an inner-product norm on Z. The examples of signal energy in a window, and squared-magnitude of the Fourier transform are special cases of this general form.

We first characterize the range of values that $||Ux||_Z$ may take: this is an interval. The optimum minimax estimate is the center of the interval.

The ellipsoid K is of the form:

$$K = \{x : (x, x) \leq 1\}$$

The set $F \cap K$, the intersection of the subspace F with the ellipsoid K, is also an ellipsoid in F:

$$F \cap K = \{x \in F : (P^{-1}x, P^{-1}x) \leq 1\}$$

where P denotes the projection operator from X onto F, and $P^{-1} : F \to X$ is the pseudo-inverse, which is positive-definite on F.

The image of $F \cap K$ in the inner-product space Z, $U(F \cap K)$, is also an ellipsoid:

$$U(F \cap K) = \{z : (P^{-1}U^{-1}z, P^{-1}U^{-1}z) \leq 1\}$$

where U^{-1} is the pseudo-inverse of z. The operator $(UP)^{-1}$ is positive definite: if $P^{-1}U^{-1}z = 0$, then $U^{-1}z = 0$, since P^{-1} is positive definite, and $UU^{-1}z = z = 0$.

The maximum value that $||Ux||_Z$ may take is the maximum of a quadratic form, subject to a quadratic constraint:

$$\max_{x \in C(y)} ||Ux||_Z = \max_{((UP)^{-1}z, (UP)^{-1}z) \leq 1} ||z + UI^{-1}y||_Z \tag{3}$$

There is a similar form for the minimum.

Quadratic maximization/minimization problems of the form (3) have been considered in [7,8]: we use this work to evaluate the range of values that $||Ux||_Z$ may take, as x varies over $C(y)$.

There are two cases to consider. When $UC(y)$ contains the origin, the minimum value that $||Ux||_Z$ may take is 0. The maximum value of $||Ux||_Z$ is found by applying the work of [8] to (3). When $UC(y)$ does not contain the origin, then both the minimum and maximum must be found.

Once the extreme values for the range of $||Ux||_Z$ have been found, the optimum minimax estimate is the average of the maximum and minimum values. It is an optimum minimax estimate because it is the point that minimizes the worst-case Z distance over all possible values of $||Ux||_Z$.

4. EXAMPLE

A simple example illustrates the differences that may arise between the optimum minimax quadratic estimate, and forming an estimate from minimax linear estimates. The problem considered is one of estimating two consecutive missing samples of a length-6 signal. In addition it is desired to estimate the sum of the squares of the two samples: the signal energy in a length-2 time window. The unknown length-6 signal is assumed to belong to an ellipsoidal class, determined by a quadratic form based on a positive definite symmetric Pascal matrix.

Figure 4 shows the interpolated signal, where the unknown time-samples at indices 3 and 4 have been estimated using optimum minimax estimation for linear functionals. Also shown are the error bars for the unknown samples: these are indicated by the dotted lines.

Figure 5 shows the image of the hypercircle in the space spanned by the two samples, $UC(y)$. It is the area bounded by ellipse B. The center of ellipse B is the intersection of the major and minor axes: it is the length-2 vector whose entries are the optimum minimax *linear* estimates. Any point in ellipse B represents a pair of possible values for time-samples 3 and 4, given the known samples at indices 1,2,5,6, and the quadratic constraint. The corresponding energy is the squared-magnitude of the length-2 position vector. Since the

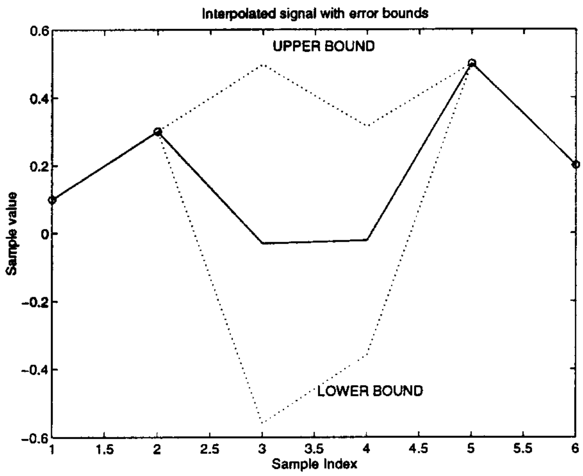

Figure 4: Interpolated samples at indices 3 and 4, with error bars

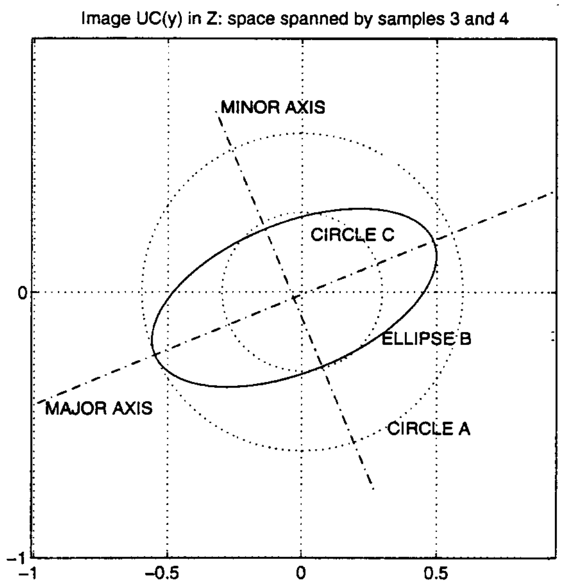

Figure 5: Ellipsoid $UC(y)$ in space spanned by samples 3 and 4

origin is contained in ellipse B, it is possible for both time-samples 3 and 4 to be simultaneously zero: thus the minimum possible energy in the time-window is zero. The maximum possible energy is achieved by the point of ellipse B that is furthest away from the origin. This is the point of tangency between circle A and ellipse B, as shown in Figure 5. Circle C has radius equal to the average of the maximum and minimum possible lengths of vectors lying in ellipse B. Its radius, 0.299, is thus the optimum minimax estimate of the square-root of the energy in the time-samples 3 and 4. This is to be contrasted with the square-root of the energy in the minimax linear estimates, 0.037, which is the length of the position vector of the center of ellipse B. These two estimates differ by nearly an order of magnitude: the percentage error, expressed as a percentage of the optimum estimate, is 87 %.

5. CONCLUSIONS AND ACKNOWLEDGEMENTS

Optimum minimax estimation of quadratic functionals has been described. A simple example shows that the estimates provided by the new procedure may differ from suboptimum answers derived from underlying linear estimates by an order of magnitude.

We would like to thank Professor Gene Golub, who brought reference [8] to our attention.

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