"ERROR SENSITIVITY ANALYSIS OF A BANDWIDTH EXPANSION TECHNIQUE IN SPECTRAL ANALYSIS"

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ABSTRACT

The paper presents an error sensitivity analysis of a bandwidth expansion technique in spectral analysis based on a noise–protected version of the Chinese Remainder Theorem. Necessary and sufficient conditions on remainder percent error and sampling period /time delay error are derived using various properties from number theory. The developed design criteria have special relevance to real–time, wide bandwidth spectral estimation in the EW passive receiver.

I. INTRODUCTION

Real–time frequency estimation over a large spectral bandwidth is a major design requirement for the modern EW passive receiver. EW frequency estimators may include the FFT, parametric methods like the covariance algorithm, or the analog IFM receiver. In the following discussion, the FFT receiver is emphasized.

Because of A/D converter sampling frequency and bit level limitations, a single FFT frequency unit cannot provide the required bandwidth or two signal dynamic range. As an alternative approach, a bank of FFT units can be operated in parallel at different sub–Nyquist sampling rates. The resulting aliasing ambiguity can be resolved and the system bandwidth expanded through a noise–protected version[1] of the Chinese Remainder Theorem (CRT).

This paper will discuss the application of the technique to real–time spectral estimation emphasizing error specification, achievable bandwidths, choice of sampling rates, time delays, and the effects of errors in the sampling period/time delay. The single frequency case will be emphasized here although the multiple frequency case is equivalent provided all the receiver unit outputs for a given frequency can be correctly grouped together[2].

II. DISCUSSION OF THE NOISE PROTECTED CRT

The Chinese Remainder Theorem provides a closed form solution for the classical simultaneous congruence problem of number theory[3]. In equation form, this problem can be stated as the solution for the integer, N, given I remainders, a_i ($i = 1,...,I$), and I relatively prime moduli, F_i,; i.e.,

$$\begin{aligned} N &= a_1 \,(\text{mod}\,[F_1]) = a_1 + K_1F_1 \\ N &= a_2 \,(\text{mod}\,[F_2]) = a_2 + K_2F_2 \\ &\vdots \\ N &= a_i \,(\text{mod}\,[F_i]) = a_i + K_iF_i \\ &\vdots \\ N &= a_I \,(\text{mod}\,[F_I]) = a_I + K_IF_I \end{aligned} \qquad (1)$$

where $0 \leq N \leq \prod_{i=1}^{I} F_i$ and where K_i are unknown integers. In engineering terms, the formulation of (1) describes a general class of aliasing problems. One well–known example is the range ambiguity problem[4] of pulse doppler radar which can be formulated, using multiple pulse repetition frequencies, as a simultaneous congruence problem in the single target case.

Because of the discontinuous nature of the K_i integers, catastrophic errors may appear in the solution of (1) when the remainders $\{a_i\}$ are in error. In reference 1, a remainder error protection level of q was introduced by defining $F_i = (4q+1)\, p_i$ as the new moduli with $\{p_i\}$ representing a set of relatively prime integers and with the expanded unambiguous bandwidth, B_μ, equal to

$$B_\mu = \frac{\prod_{i=1}^{I} F_i}{(4q+1)^{I-1}} = p_1p_2...F_I \qquad (2)$$

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To illustrate, let us consider the two receiver case (I = 2). To avoid errors $\geq$ min (F_i, F_j) with noise perturbed remainders, $\hat{\alpha}_i$, $\hat{\alpha}_j$, the expressions $\hat{\alpha}_i + K_iF_i$ and $\hat{\alpha}_j + K_jF_j$ must be within 2q where the remainder perturbation magnitudes are $\leq$ q and K_i,K_j are the same integers as in the noise free case (equation 1). The $F_i = (4q+1)\ p_i$ (i = 1,2) condition ensures that the following equality holds

$$|(\bar{K}_i - K_i)\ F_i - (\bar{K}_j - K_j)\ F_j|_{min} = 4q+1 \quad (3)$$

where $\bar{K}_i$, $\bar{K}_j$ represent incorrect integers and K_i, $\bar{K}_i < p_j$, $K_j,\bar{K}_j < p_i$. Condition (3) establishes a sufficient distance so that the following inequality is true

$$|\hat{\alpha}_i - \hat{\alpha}_j + K_iF_i - K_jF_j| < 2q+1 \quad (4)$$

for only the correct values of K_i, K_j for noise perturbations $\leq$ q. In the hardware implmentation, the F_i's refer to the bandwidth (MHz) or sampling rate of the corresponding FFT receiver and $\tau_i = 1/F_i$ represents the sampling period (µsec.). The two receiver case of (3) and (4) can be extended to the general I receiver case since it can be shown[5] that (1) can be solved as a sequence of two receiver solutions. For general I, it is possible to free the choice of τ_i from the value of F_i by setting $F_i\tau_i = C$ where C is any real number and F_i is now a dimensionless integer. The $F_i\tau_i = C$ constraint preserves the number theory based noise protection since it is equivalent to multiplying both sides of (4) by $(1/\tau_i/F_i) = 1/C$ which is the new frequency unit. From the constraint,

$$F_1\tau_1 = F_2\tau_2 = F_3\tau_3 = \cdots = C \ \ (\text{secs}) \quad (5)$$

we can divide (5) by (4q+1) to get

$$p_1\tau_1 = p_2\tau_2 = p_3\tau_3 = \cdots = \frac{C}{4q+1} \ \ (\text{secs}). \quad (6)$$

With a choice of τ_1, the other time delays, τ_i, can be calculated as

$$\tau_i = \left[\frac{p_1}{p_i}\right] \tau_1; \quad i = 2,3,\ldots,I \quad (7)$$

which permits the design of implementable τ_i's in either an analog or digital configuration. For τ_i's related as in (7), the unambiguous bandwidth can now be written from (2) as

$$B_\mu = \left[\frac{1}{\tau_1}\right] p_2\, p_3\, p_4 \cdots p_I. \quad (8)$$

III. ERROR SPECIFICATION

For a given S/N ratio and finite word length, 2^b (b bits of quantization), the frequency accuracy, δf, of an FFT receiver can be stated either as an absolute frequency error, $|\Delta f|$, or as a ratio of the unambiguous bandwidth, $1/\tau_i$; i.e., $(|\Delta f|)\ \tau_i$. For the absolute error specification, the q (dimensionless) can be specified as

$$q \geq \text{Int}\,\{C|\Delta f|\} + 1 \quad (9)$$

with overall bandwidth increasing with increasing I as in (8) with $C = \tau_iF_i$. In the real-time EW passive receiver, however, the outputs of the FFT receivers are usually represented by word lengths of b bits. As a result, a more natural error specification, δ, on output frequency should be written as a ratio or percent error as

$$\delta = \frac{m_i}{2^{b_i}}; \quad i = 1,\ldots,I, \quad (10)$$

where m_i is the number of quantization levels of error, and b_i is the number of bits of quantization. The corresponding necessary and sufficient condition for complete noise protection can then be written as

$$\frac{m_i}{2^{b_i}} < \frac{q}{F_i} = \frac{q}{(4q+1)\ p_i} < \frac{1}{(4+1/q)\ p_i} < \frac{1}{4p_i}. \quad (11)$$

As an illustrative example, consider the case where I = 3, $m_1 = m_2 = m_3 = 3$, $b_1 = b_2 = b_3 = 6$, and $p_1 = 2$, $p_2 = 3$, $p_3 = 5 = \bar{p}$ where $\bar{p}$ is the largest of the relatively prime integers. In this example, $m/2^b = 0.04875$ which is less than $1/4p_i$, for all i, which then ensures complete noise protection. Setting $m/2^b$ equal to the $1/(4+1/q)\ \bar{p}$ of (11) implies a q of 4 which generates $F_1 = 34$, $F_2 = 51$, and $F_3 = 85$. For an implementable τ_2 of 25 n.s., (7) gives values of $\tau_1 = 37.5$ n.s. and $\tau_3 = 15.00$ n.s. with an overall bandwidth from (8) of 400 MHz. A significant bandwidth expansion has occurred compared to the $1/\tau_i$ single receiver bandwidths of 26.6 MHz, 40 MHz, and 66 MHz respectively. The further expansion of bandwidth requires a fourth receiver and a minimum $\bar{p}$ of 7 which will result in a violation of the criterion of (11) provided the $m_i/2^{b_i}$ is not reduced below the value of 0.04875. It follows that simply increasing I in order to increase B_μ may not be successful since an increase in I represents an increase in $\bar{p}$ and a corresponding decrease in the maximum acceptable $(m_i/2^{b_i})$. This interesting effect is due to the increase in frequency error or required q value that is implied by an increasing $1/\tau_I$ (or increasing I) value for the percent error specification. The constraints of (9) and (11) have been completely verified in simulation.

IV. EFFECT OF SAMPLING PERIOD OR TIME DELAY ERROR

In the following discussion, τ_1 can refer to either the sampling period of an FFT receiver unit or the analog time delay of a conventional IFM receiver.

Let us assume an error of $\Delta\tau_1$ in the sampling period or time delay value, τ_1. With f equaling a frequency within B_μ, we can write the perturbation in F_1, from (5) as

$$\Delta F_1 = \Delta\left[\frac{C^{-1}}{\tau_1}\right] = C^{-1}\left[-\frac{\Delta\tau_1}{\tau_1^2}\right] \qquad (12)$$

which results in the following equality,

$$\begin{aligned} f &= (\alpha + K_1F_1)\left[\frac{C^{-1}}{\tau_1}\right] \\ &= \left[\beta + \bar{K}_1(F_1 + \Delta F_1)\right]\left[\frac{C^{-1}}{\tau_1}\right] \end{aligned} \qquad (13)$$

where α, K_1 and β, $\bar{K}_1$ correspond to unperturbed/perturbed values. Transposing and dividing by C^{-1}/τ_1, we can write the remainder perturbation as

$$(\beta - \alpha) = (K_1F_1 - \bar{K}_1F_1) - \bar{K}_1\,\Delta F. \qquad (14)$$

Assuming that the error perturbation is small, we can take $K_1 = \bar{K}_1$ which leads to

$$|\beta - \alpha| = K_1\,|\Delta F_1| = K_1\left|\frac{\Delta\tau_1}{\tau_1^2}\right|. \qquad (15)$$

Clearly, the perturbation is greatest when $K_1 = (K_1)_{max} = p_2p_3 \ldots p_I$ where I is the number of receivers used; $|\beta-\alpha|_{max}$ can then be written, in terms of overall bandwidth, B_μ, using (8), as

$$|\beta - \alpha|_{max} = B_\mu\left|\frac{\Delta\tau_1}{\tau_1}\right|. \qquad (16)$$

Using (11), we can now specify the following condition for error free operation for I receivers using the ratio or percent error specification,

$$\frac{|\beta-\alpha|_{max}}{F_i} < \frac{q}{F_i} < \frac{1}{(4+1/q)\ p_i} < \frac{1}{4p_i};\quad i = 1,\ldots,I. \qquad (17)$$

In terms of (16), expression (17) can be immediately rewritten as

$$\frac{B_\mu}{F_i}\left|\frac{\Delta\tau_i}{\tau_i}\right| = p_1p_2 \ldots p_{i-1} \cdot p_{i+1} \ldots p_i\left|\frac{\Delta\tau_i}{\tau_i}\right| < \frac{1}{4p_i} \qquad (18)$$

which can be compactly written as

$$\left|\frac{\Delta\tau_i}{\tau_i}\right| < \frac{1}{4\prod_{i=1}^{I} p_i} \qquad (19)$$

which imposes a fundamental limit on the acceptable percent error in the time period (time delay) allowed in the practical implementation. It follows that (19) and (11) together form necessary and sufficient conditions for error–free performance.

To illustrate, a three receiver case was considered with $p_1 = 2$, $p_2 = 3$, $p_3 = 7$, which from (19) leads to the inequality

$$\left|\frac{\Delta\tau_i}{\tau_i}\right| < \frac{1}{4p_1p_2p_3} = 0.0060 \qquad (20)$$

which implies that time period/delays must be accurately set to within 0.60% to avoid catastrophic error. The bound of (20) appears to be a tight one since simulation results give error-free results for 0.5% accuracy and complete failure of the ambiguity algorithm for a 0.75% accuracy level.

V. CONCLUSIONS

An error sensitivity analysis was performed for a bandwidth expansion technique based on a noise-protected version of the Chinese Remainder Theorem. Necessary and sufficient conditions for error–free performance were derived for maximum absolute and percentage perturbations in the receiver remainders and also for permissible percentage error in the sampling period (digital methods) and analog delay times (IFM receiver). The design rules apply to a general number of receivers and also to the multiple frequency case.[2]

REFERENCES

[1] McCormick, W. S., Tsui, J. B. Y., Bakke, V. L., "A noise insensitive solution to an ambiguity problem in spectral estimation," IEEE Trans. Aerospace and Electronic Systems, vol. 25, no. 5, pp. 729–732, 1989.

[2] McCormick, W. S., Miller, D. F., Tsui, J. B. Y., "Resolution of a 2π ambiguity problem in

multiple frequency spectral estimation," IEEE Trans. Aerospace and Electronic Systems, Vol. 31, No. 1, pp. 1–7, January 1995.

[3] Kirch, A. M., Elementary Number Theory: A Computer Approach: Intext Educational Publishers, 1974.

[4] Addison, E. R. and Frost, E. L., "Resolution of ambiguous radar measurements using a floating bin correlator," In Proc. IEEE 1982 Radar Conference, pp. 144–148.

[5] Miller, D. F. and McCormick, W. S., "Resolving 2π ambiguous multiple frequency estimates using relaxed simultaneous congruences," submitted for publication.