# FAST ALGORITHMS FOR SYSTEMS OF EQUATIONS IN WAVELET-BASED SOLUTION OF INTEGRAL EQUATIONS WITH TOEPLITZ KERNELS

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## ABSTRACT

The wavelet transform is applied to integral equations with Toeplitz kernels. Such integral equations arise in inverse scattering and linear least-squares estimation. The result is a system of equations with block-slanted-Toeplitz structure. In previous approaches, this linear system was sparsified by neglecting all entries below some threshold. However, in inverse scattering, the Toeplitz kernel may not be a rapidly decreasing function due to reflections from great depths. In this case, neglecting entries below a threshold will not work since the system matrix is ill-conditioned. We use the different approach of exploiting the block-slanted-Toeplitz structure to obtain fast algorithms similar to the multichannel Levinson and Schur algorithms. Since it is exact to within the wavelet-basis approximation, this different approach should prove to be a valuable alternative to the approximate approach of sparsification in cases when the latter does not work.

## 1. INTRODUCTION

The mathematical inverse problem of reconstructing a one-dimensional continuous layered medium from its impulse reflection response has many applications in many fields. These include reflection seismology [1], acoustic measurement of the shape of the human vocal tract, and the synthesis of nonuniform transmission lines. All of these problems can be formulated as the nonlinear problem of reconstructing a spatially-varying reflectivity function $r(x)$ in the two component wave system (1) [2], from a temporally-varying impulse reflection response function $k(t)$. Note the problem is nonlinear due to multiple scattering in the wave system; these effects are included here, unlike some methods which ignore multiple scattering (the Born approximation).

We apply the wavelet transform to the Krein integral equation (4) [3] of inverse scattering. This is also the Wiener-Hopf integral equation for computing the linear least-squares estimation filter for a stationary random process, so our results are directly applicable to that problem.

Previous methods applying wavelet transforms to the solution of integral equations resulted in linear systems of equations in which entries below a threshold were neglected to obtain a sparse system. This works well if the wavelet representation of the integral equation kernel is rapidly decreasing, which generally requires that the kernel itself be a rapidly decreasing function such as the covariance of a first-order Markov random process. However, in inverse scattering the kernel is the reflection response of the medium to a probing impulse or impulsive plane wave. Such a kernel is not a rapidly decreasing function since primary reflections from great depths will keep the function from dying out. Hence, sparsification will often not work since the system is ill-conditioned and neglecting entries will significantly alter the solution.

We propose to use a fast algorithm to exploit the block-slanted-Toeplitz structure of the linear system of equations (9). This algorithm solves the system exactly, without the approximation inherent in sparsification. It has a form similar to the multichannel Levinson and Schur algorithms. Rather than present the derivation in this limited space, we show quickly why this is so.

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## 2. REVIEW OF INVERSE SCATTERING AND THE WAVELET TRANSFORM

### 2.1. The 1-D Inverse Scattering Problem

Let $x$ be a spatial variable and $t$ be time. Scattering media are described by the two-component wave system [2]

$$\frac{\partial}{\partial x}\begin{bmatrix} d(x,t) \\ u(x,t) \end{bmatrix} = \begin{bmatrix} -\frac{\partial}{\partial t} & -r(x) \\ -r(x) & \frac{\partial}{\partial t} \end{bmatrix} \begin{bmatrix} d(x,t) \\ u(x,t) \end{bmatrix} \qquad (1)$$

where the reflectivity function $r(x)$ characterizes the scattering medium. If $r(x) = 0$, then $d(x,t) = d(x-t)$ and $u(x,t) = u(x+t)$. Thus $d(x,t)$ and $u(x,t)$ can be interpreted as Downgoing and Upgoing waves, scattered into each other at depth $x$ by the reflectivity function $r(x)$.

The scattering medium is assumed to have finite extent in $x$; without loss of generality, this extent is scaled to $0 < x < 1$. The boundary conditions are a radiation condition at $x = 1$ ($d(x,t) = d(x-t)$ and $u(x,t) = 0$ for $x > 1$) and a free surface at $x = 0$ ($d(0,t) = u(0,t)$, excluding sources). The free surface implies that an upgoing wave at $x = 0$ is simply reflected into a downgoing wave.

This scattering medium is probed with an impulsive plane wave $\delta(t-x)$, which propagates downward into the medium in increasing depth $x$ as time $t$ increases. The reflection response $k(t)$ of the medium to this impulse is measured at $x = 0$. This amounts to initializing (1) with

$$d(0,t) = \delta(t) + k(t); \quad u(0,t) = k(t). \qquad (2)$$

The inverse scattering problem is to compute the reflectivity function $r(x)$ from the impulse reflection response $k(t)$.

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Several types of inverse problems can be formulated as the above problem. For example, if the scattering medium is a continuously-layered acoustic medium with constant wave speed and varying density $\rho(x)$, then the problem of reconstructing $\rho(x)$ from the reflection response of the medium to an impulsive plane wave $\delta(t-x)$ can be formulated as (1) by defining $r(x) = (d/dx)\log\sqrt{\rho(x)}$ and

$$d(x,t), u(x,t) = \frac{p(x,t)}{\sqrt{\rho(x)}} \pm \sqrt{\rho(x)}v(x,t) \qquad (3)$$

where $p(x,t)$ is pressure in the medium and $v(x,t)$ is velocity of the medium. In other applications $\rho(x)$ is replaced with local impedance of a nonuniform transmission line or cross-sectional area of the human vocal tract.

### 2.2. Solution to 1-D Inverse Scattering Problem

The inverse scattering problem can be solved by solving the Krein integral equation (note the Toeplitz structure in the kernel $k(|x-t|)$) [3]

$$k(x-t) = h(x,t) + \int_{-x}^{x} h(x,z)k(|z-t|)dz; \quad |t| \le x; \quad 0 \le x \le 1 \qquad (4)$$

for $h(x,t)$. $r(x)$ can then be computed from $h(x,t)$ using

$$r(x) = 2h(x,-x). \qquad (5)$$

We recognize (4) as the Wiener-Hopf integral equation for computing the linear least-squares filter $h(x,t)$ for estimating a zero-mean wide-sense stationary random process, with covariance function $k(|x-t|)$, at time $x$ from noisy observations, with additive white noise, measured over the interval $-x < t < x$. (5) then merely states the well-known result of linear prediction that the reflectivity function (continuous reflection coefficient) equals the filter weight at the far end of the interval of observation.

### 2.3. Discrete Orthonormal Wavelet Transforms

The discrete orthonormal wavelet transform $F(m,n)$ of a continuous square-integrable function $f(x)$ is

$$F(m,n) = \int_{-\infty}^{\infty} f(x)2^{m/2}\phi(2^m x - n)dx \qquad (6a)$$

$$f(x) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} F(m,n)2^{m/2}\phi(2^m x - n) \qquad (6b)$$

where $\phi(x)$ is the wavelet basis function. $\phi(x)$ is orthogonal (in the sense of the usual $L^2$ inner product) to its scalings $\phi(2^m x)$ (dilations for $m<0$; compressions for $m>0$) and to the translations $\phi(2^m x - n)$ of its scalings, and the set of all scalings and translations $\{2^{m/2}\phi(2^m x - n); m, n \in \textit{integers}\}$ forms a complete orthonormal set.

## 3. WAVELET REPRESENTATION OF THE KREIN INTEGRAL EQUATION

### 3.1. Wavelet Expansions

First, we make a minor change in the Krein integral equation (4). Define

$$k'(x) = 2k(2x); \quad h'(x,t) = 2h(x, 2t-x); \qquad (7a)$$

$$t' = (x+t)/2; \quad z' = (x+z)/2. \qquad (7b)$$

This merely changes the interval $-x \le t \le x$ to the interval $0 \le t' \le x$, for the ranges of integration and validity.

Let $K'(m_3,n_3)$ be the wavelet transform (6a) of $k'(x)$ and $H'(m_1,n_1,m_2,n_2)$ be the double wavelet transform of $h'(x,t)$ where $x$ maps to $(m_1,n_1)$ and $t$ maps to $(m_2,n_2)$.

Also define the discrete function

$$E(m_1,n_1,m_2,n_2,m_3,n_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2^{m_1/2}\phi(2^{m_1}x - n_1)$$

$$\times 2^{m_2/2}\phi(2^{m_2}y - n_2)2^{m_3/2}\phi(2^{m_3}(x-y) - n_3)\,dx\,dy$$

$$= E(m_1,m_2,m_3,n_3,(2^{-m_1}n_1 - 2^{-m_2}n_2)). \qquad (8)$$

$E(m_1,m_2,m_3,n_3,(2^{-m_1}n_1 - 2^{-m_2}n_2))$ has slanted-Toeplitz structure, defined as being a function not of $n_1$ and $n_2$ separately, but a function of only their weighted difference $2^{-m_1}n_1 - 2^{-m_2}n_2$. This is illustrated in the matrix below.

### 3.2. Linear System of Equations

Inserting wavelet expansions of $k'(x)$ and $h'(x,t)$ into the modified Krein integral equation and applying the wavelet transform (6a) in $x$ and $t$ yields

$$K''(m_1,m_2,(2^{-m_1}n_1 - 2^{-m_2}n_2)) = H'(m_1,n_1,m_2,n_2)$$

$$+ \sum_{m_3=0}^{M}\sum_{n_3} H'(m_1,n_1,m_3,n_3)K''(m_3,m_2,(2^{-m_3}n_3 - 2^{-m_2}n_2));$$

$$0 \le n_1 \le 2^{m_1} - 1; \quad 0 \le 2^{-m_2}n_2 \le 2^{-m_1}n_1 - 1, \qquad (9)$$

where $K''(\cdot)$ is defined from (8) as

$$K''(m_1,m_2,(2^{-m_1}n_1 - 2^{-m_2}n_2))$$

$$= \sum_{m_3=0}^{M}\sum_{n_3=0}^{2^{m_3}-1} E(m_1,m_2,m_3,n_3,(2^{-m_1}n_1 - 2^{-m_2}n_2))K'(m_3,n_3), \qquad (10)$$

and $M$ is the (arbitrarily large) finest scale used.

The system matrix specified by $K''(\cdot)$ in the system of equations (9) has block-slanted-Toeplitz structure, in that in the $(m_1,m_2)^{th}$ block the elements $(n_1,n_2)$ are equal for constant values of $2^{-m_1}n_1 - 2^{-m_2}n_2$, i.e., along diagonals of slope $-2^{(m_1-m_2)}$. Note that matrix coordinates start at $(0,0)$ not $(1,1)$. The system matrix has the following structure (letters denote equal entries, excluding symmetry):

$$\begin{bmatrix} * & \begin{bmatrix} * & * \end{bmatrix} & \begin{bmatrix} * & * & * & * \end{bmatrix} & \cdots \\ \begin{bmatrix} * \\ * \end{bmatrix} & \begin{bmatrix} a & * \\ * & a \end{bmatrix} & \begin{bmatrix} b & f & * & * \\ * & * & b & f \end{bmatrix} & \cdots \\ \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix} & \begin{bmatrix} b & * \\ f & * \\ * & b \\ * & f \end{bmatrix} & \begin{bmatrix} c & d & e & * \\ d & c & d & e \\ e & d & c & d \\ * & e & d & c \end{bmatrix} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad (11)$$

## 4. FAST ALGORITHM

### 4.1. Matrix Rearrangements

Examination of the form (11) of the system matrix (10) in (9) shows that (9) can be rewritten as a set of linear block-slanted-Toeplitz systems

$$\begin{bmatrix} a & \begin{bmatrix} n & o \end{bmatrix} & \begin{bmatrix} p & q & r & s \end{bmatrix} \\ \begin{bmatrix} n \\ o \end{bmatrix} & \begin{bmatrix} b & c \\ c & b \end{bmatrix} & \begin{bmatrix} h & i & j & k \\ l & m & h & i \end{bmatrix} \\ \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} & \begin{bmatrix} h & l \\ i & m \\ j & h \\ k & i \end{bmatrix} & \begin{bmatrix} d & e & f & g \\ e & d & e & f \\ f & e & d & e \\ g & f & e & d \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix}$$

where the system matrix is $I+K''$, the vector $x$ is a column of the matrix $H'^T$, and the vector $y$ is a column of the matrix $K''$. This can then be rewritten as

$$\begin{bmatrix} a & 0 & 0 & 0 & n & 0 & o & 0 & p & q & r & s \\ 0 & a & 0 & 0 & 0 & n & 0 & o & 0 & p & q & r \\ 0 & 0 & a & 0 & 0 & 0 & n & 0 & 0 & 0 & p & q \\ 0 & 0 & 0 & a & 0 & 0 & 0 & n & 0 & 0 & 0 & p \\ n & 0 & 0 & 0 & b & 0 & c & 0 & h & i & j & k \\ 0 & n & 0 & 0 & 0 & b & 0 & c & m & h & i & j \\ o & 0 & n & 0 & c & 0 & b & 0 & l & m & h & i \\ 0 & o & 0 & n & 0 & c & 0 & b & 0 & l & m & h \\ p & 0 & 0 & 0 & h & m & l & 0 & d & e & f & g \\ q & p & 0 & 0 & i & h & m & l & e & d & e & f \\ r & q & p & 0 & j & i & h & m & f & e & d & e \\ s & r & q & p & k & j & i & h & g & f & e & d \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \\ x_2 \\ 0 \\ x_3 \\ 0 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} y_1 \\ * \\ * \\ * \\ y_2 \\ * \\ y_3 \\ * \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} \quad (12)$$

Moreover, these systems of equations with Toeplitz blocks can be rearranged as block-Toeplitz systems. In this case, (12) becomes

$$\begin{bmatrix} a & n & p & 0 & 0 & q & 0 & o & r & 0 & 0 & s \\ n & b & h & 0 & 0 & i & 0 & c & j & 0 & 0 & k \\ p & h & d & 0 & m & e & 0 & l & f & 0 & 0 & g \\ 0 & 0 & 0 & a & n & p & 0 & 0 & q & 0 & o & r \\ 0 & 0 & m & n & b & h & 0 & 0 & i & 0 & c & j \\ q & i & e & p & h & d & 0 & m & e & 0 & l & f \\ 0 & 0 & 0 & 0 & 0 & 0 & a & n & p & 0 & 0 & q \\ o & c & l & 0 & 0 & m & n & b & h & 0 & 0 & i \\ r & j & f & q & i & e & p & h & d & 0 & m & e \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & n & p \\ 0 & 0 & 0 & o & c & l & 0 & 0 & m & n & b & h \\ s & k & g & r & j & f & q & i & e & p & h & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ 0 \\ 0 \\ x_5 \\ 0 \\ x_3 \\ x_6 \\ 0 \\ 0 \\ x_7 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_4 \\ * \\ * \\ y_5 \\ * \\ y_3 \\ y_6 \\ * \\ * \\ y_7 \end{bmatrix} \quad (13)$$

To solve for $H'^T$, the inverse of the block-slanted-Toeplitz matrix must be computed in terms of the inverse of the block-Toeplitz matrix in (13) and then applied to the matrix of $y$ vectors (i.e., $K''$)). The procedure is as follows.

### 4.2. Multichannel Levinson Algorithm

First, the inverse of the block-Toeplitz system in (13) is computed by running the multichannel Levinson algorithm

$$\begin{bmatrix} A_n(z) \\ B_n(z) \end{bmatrix} = \begin{bmatrix} I & zK_n^1 \\ K_n^2 & zI \end{bmatrix} \begin{bmatrix} A_{n-1}(z) \\ B_{n-1}(z) \end{bmatrix},$$

$$K_n^1 = -\textstyle\sum_{i=0}^{n-1} A_{n-1,i} R_{n-i}^H (P_{n-1}^2)^{-1},$$

$$K_n^2 = -\left(\textstyle\sum_{i=0}^{n-1} A_{n-1,i} R_{n-i}^H\right)^H (P_{n-1}^1)^{-1}, \quad (14)$$

$$P_n^1 = (I - K_n^1 K_n^2) P_{n-1}^1,$$

$$P_n^2 = (I - K_n^2 K_n^1) P_{n-1}^2.$$

The algorithm is initialized with $A_{0,0} = B_{0,0} = I$ and $P_0^1 = P_0^2 = R_0$, where

$$A_n(z) = \sum_{i=0}^{n} A_{n,i} z^i, \qquad B_n(z) = \sum_{i=0}^{n} B_{n,i} z^i.$$

This recursively solves in increasing $n$ the system

$$\begin{bmatrix} R_0 & R_1^H & \cdots & R_n^H \\ R_1 & R_0 & & \vdots \\ \vdots & & \ddots & \\ R_n & \cdots & & R_0 \end{bmatrix} \begin{bmatrix} A_{n,0}^H & B_{n,0}^H \\ \vdots & \vdots \\ A_{n,n}^H & B_{n,n}^H \end{bmatrix} = \begin{bmatrix} P_n^1 & 0 \\ 0 & \vdots \\ \vdots & 0 \\ 0 & P_n^2 \end{bmatrix}$$

where the $R_i$ are the blocks in (13).

The Schur algorithm can be run simultaneously with the Levinson recursions for more efficient computation of $K_n^1$ and $K_n^2$. This avoids the summations in (14). The inverse $C^{-1}$ of the block-Toeplitz matrix is then computed using the multichannel Gohberg-Semencul formula

$$C^{-1} = T^T(A_n^H)T(A_n^H) - T^T(B_n^H Z)T(B_n^H Z)$$

$$B_n^H Z = \begin{bmatrix} B_{n,1}^H \cdots B_{n,n}^H 0 \end{bmatrix}$$

$$T(A_n^H) = \begin{bmatrix} I & 0 & \cdots & 0 \\ A_{n,1}^H & \ddots & & \vdots \\ \vdots & \ddots & & \\ A_{n,n}^H & & A_{n,1}^H & I \end{bmatrix} \quad (15)$$

### 4.3. Inverses of Submatrices using Outer Products

Second, the inverse $B^{-1}$ of the Toeplitz blocks matrix in (12) is computed from $C^{-1}$ by noting that the block-Toeplitz matrix $C$ in (13) can be written as $C = E_k \cdots E_1 B E_1 \cdots E_k$ where each $E_i$ is a self-inverse elementary matrix that performs either a row or a column exchange. Then we have

$$B^{-1} = E_1 \cdots E_k C^{-1} E_k \cdots E_1. \quad (16)$$

Third, the inverse of the block-slanted-Toeplitz matrix $D^{-1}$ is computed from the inverse of the Toeplitz blocks matrix $B^{-1}$. Note that $B$ can be rearranged into

$$A = E_j \cdots E_1 B E_1 \cdots E_j = \left[\begin{array}{c|c} D & * \\ \hline * & * \end{array}\right]$$

where $A = E_j \cdots E_1 B E_1 \cdots E_j$ and thus

$$A^{-1} = E_j \cdots E_1 B^{-1} E_1 \cdots E_j. \tag{17}$$

Also, $A^{-1}$ can be expressed as

$$A^{-1} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]^{-1} = \left[\begin{array}{c|c} A_{11}^{-1} + [A^{-1}]_{12} [A^{-1}]_{22}^{-1} [A^{-1}]_{21} & [A^{-1}]_{12} \\ \hline [A^{-1}]_{21} & [A^{-1}]_{22} \end{array}\right] \tag{18}$$

where $A_{11}$ is the $nxn$ upper left submatrix of $A$ and $A_{22}$ is an $M \times M$ matrix. Thus $A_{11}^{-1}$ can be computed as

$$A_{11}^{-1} = [A^{-1}]_{11} - [A^{-1}]_{12} [A^{-1}]_{22}^{-1} [A^{-1}]_{21}. \tag{19}$$

Repeat this process until $A_{11} = D$.

Finally, the solution to the original block-slanted-Toeplitz system (11) is $H'^T = D^{-1} K''$.

### 4.4. Overall Procedure

To reconstruct the wavelet transform $R(m, n)$ of $r(x)$ from the wavelet transform $K(m, n)$ of $k(t)$:

1. Compute $K''(\cdot)$ from $k(t)$ using (6), (7) and (10);
2. Solve the linear system of equations (11) for $H'$ using (14)-(18);
3. Compute $R(m, n)$ from $H'(\cdot)$ using

$$R(m_1, n_1) = \sum_{m_2=0}^{M} H'(m_1, n_1, m_2, n_2 = 0) 2^{m_2/2} \phi(0^+) \tag{20}$$

relating the wavelet transforms $K(m, n)$ and $R(m, n)$.

If the Krein integral equation is discretized with $\Delta = 2^{-M}$, the resulting system of equations has size $2^M$. From (11), the size of the system (9) is ($M$=largest scale used)

$$1 + 2 + 4 + 8 + \ldots + 2^M = 2^{M+1} - 1 \tag{21}$$

so (9) is only twice as large as the simple discretization.

Direct computation of $H'^T$ would require $\mathcal{O}(2^{4M})$ operations. The above fast algorithm, which uses the fast multichannel Levinson algorithm to effectively invert the larger, rearranged matrix $A$ and then uses (19) to compute the inverse of a submatrix by computing outer products, requires $\mathcal{O}(2^{3M})$ operations. Note that the block size is $M \times M$ and $M << 2^M$, so the presence of blocks, and particularly the $M \times M$ matrix inversion required in (19), is not significant computationally.

## 5. NUMERICAL EXAMPLE

We have successfully used this procedure to solve the Krein integral equation, and thus the inverse scattering problem. The reflectivity function $r(x) = -1/(x+2)$ produces a free-surface reflection response $k(t) = -(1/2)e^{-t}$ for $t > 0$, so that the kernel of the Krein integral equation is $\delta(t) + k(|t|) = \delta(t) - (1/2)e^{-|t|}$. Using $M = 4$ and a Haar basis resulted in the $h(x, t)$ shown; the slice $r(x) = 2h(x, -x) = h'(x, 0)$ is also shown, and clearly matches the actual reflectivity. Using a Daubechies basis gave smoother results.

## 6. REFERENCES


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