STRUCTURED TOTAL LEAST NORM METHOD FOR TOEPLITZ PROBLEMS

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ABSTRACT

The Total Least Squares (TLS) method for solving an overdetermined system $Ax \approx b$ is a generalization of the Least Squares (LS) method, and it minimizes $||[E|r]||_F$ so that $(b+r) \in Range(A+E)$, given $A \in \mathbf{R}^{m\times n}$, with $m \geq n$ and $b \in \mathbf{R}^{m\times 1}$. The commonly used TLS algorithm is based on the singular value decomposition (SVD) of $[A\,|\,b]$. However, in applications where the matrix A has a special structure, the SVD based methods may not always be appropriate, since they do not preserve the structure. Recently, a new formulation, called Structured Total Least Norm (STLN), and algorithm for computing solutions have been developed. STLN preserves any affine structure of A or $[A\mid b]$, and can minimize error in the discrete L_p norm, where $p = 1, 2$ or ∞. In this paper, we study the STLN method for problems in which the perturbation matrix E or $[E|r]$ keeps the Toeplitz structure like the data matrix A or $[A\mid b]$. These structures occur in many problems such as deconvolution, transfer function modeling and linear prediction problems. In particular, STLN methods with L_1 and L_2 norms are compared with the LS and TLS methods and shown to improve the accuracy of the solutions significantly. When there is an outlier in the data, the STLN method with L_1 norm is shown to produce solutions that are essentially not affected by the ourlier.

1. FORMULATION OF STRUCTURED TOTAL LEAST NORM (STLN) PROBLEMS

An important data fitting technique frequently adopted in signal processing applications for solving an overdetermined system of linear equations $Ax \approx b$ is the Total Least Squares (TLS) method [2, 8]. The TLS problem can be stated as that of finding E and x, such that

$$\min_{E,x} ||[E\,|\,r]||_F, \tag{1}$$

where $r = b-(A+E)x$, given $A \in \mathbf{R}^{m\times n}$, with $m \geq n$, and $b \in R^{m\times 1}$. This problem allows the possibility of error in the elements of a given matrix A, so that the modified matrix is given by $A+E$, where E is an error matrix to be determined. The generally used computational method for solving TLS is based on the singular value decomposition (SVD) of $[A\,|\,b]$ [2, 8]. In applications where the matrix A has a special structure, the SVD based methods may not always be appropriate, since they do not preserve the special structure. In fact the matrix E obtained from the SVD will typically be dense, with no structure, even when A is structured.

Recently, a new formulation, called Structured Total Least Norm (STLN), and algorithm for computing solutions have been developed [7]. STLN preserves any affine structure of A and can minimize the error in the discrete L_p norm, where $p = 1, 2$ or ∞ [7]. A theoretical justification and computational testing of STLN algorithm confirm that it is an efficient method for solving problems where A or $[A\,|\,b]$ has a special structure, or where errors can occur only in some of the elements of A or $[A\,|\,b]$.

In this paper, we study the STLN method for problems in which the perturbation matrix E or $[E|r]$ keeps the Toeplitz structure like the data matrix A or $[A\,|\,b]$. In many applications in signal processing and system identification such as deconvolution, transfer function modeling and linear prediction problems, the matrix A or $[A|b]$ has Toeplitz or Hankel structure. In particular, STLN methods with L_1 and L_2 norms are compared with the LS and TLS methods and shown to improve the accuracy of the solutions significantly. When there is an outlier in the data, the STLN method with L_1 norm is shown to produce solutions that are essentially not affected by the ourlier. The presented results for Toeplitz structure also hold for Hankel structure, since

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Hankel matrices simply transform to Toeplitz matrices by permutations.

In STLN, a vector $\alpha \in \mathbf{R}^{q\times 1}$ ($q \leq mn$) is used to represent the corresponding elements of the error matrix $E \in \mathbf{R}^{m\times n}$. If many elements of E must have the same value, then q is the number of *different* such elements. The vector α and the matrix E are equivalent, in the sense that given E, α is known, and vice versa. Now, the residual vector $r = b-(A+E)x$, is a function of (α, x). Let $D \in \mathbf{R}^{q\times q}$ be a diagonal matrix that accounts for the repetition of elements of α in the matrix E. Then the STLN problem can be stated as:

$$\min_{\alpha,x} \left\| \begin{matrix} r(\alpha,x) \\ D\alpha \end{matrix} \right\|_p , \qquad (2)$$

where $\|\cdot\|_p$ is the vector p-norm, for $p = 1, 2$, or ∞. For $p = 2$, and a suitable choice for D, problem (2) is equivalent to the TLS problem (1), with the additional requirement that the structure of A must be preserved by $A + E$.

2. STLN FOR TOEPLITZ MATRICES

The STLN formulation for affine structured problems and an iterative algorithm have been described and justified in previous papers [6, 7]. Here, we summarize how it can be modified to handle Toeplitz structures in $[A|b]$. For example, in the LS linear prediction problem [3], we need to solve

$$\min_x \|Ax - b\|_2 \qquad (3)$$

where $A \in \mathbf{R}^{m\times n}$, $m \geq n$ and either $[A \mid b]$ or $[b \mid A]$ is Toeplitz . We first show how to modify the STLN algorithm so that it can treat possible errors in some or all elements of b in general.

We introduce a vector β representing possible errors in selected elements of b. This is similar to α representing errors in A. Suppose different errors can occur in l($\leq m$) elements of b. The error vector $\beta \in \mathbf{R}^l$ represents the error in b. The relation between β and b is given by a matrix $F \in \mathbf{R}^{m\times l}$, so that the error in b is the same as $F\beta$. The (i,j)th element of F is one if β_j is the error in b_i, otherwise, it is zero. Initially, E, α and β are all zero, and the new residual $\hat{r} = r = b - Ax$. In general,

$$\hat{r} = \hat{r}(\alpha,\beta,x) = (b + F\beta) - (A+E)x = r + F\beta.$$

In Toeplitz matrices $q\,(\leq m+n-1)$ elements of A are subject to error.

For the linear prediction problem with $[A|b]$ or $[b|A]$ Toeplitz and all $m + n$ diagonals are subject to errors,

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Algorithm STLN-LPR

Input – A Total Least Norm problem (2), with specified matrices A, D, vector b, and tolerance ϵ, such that $[A \mid b]$ is Toeplitz.
Output – Error matrix E, error β_1, and vector x, such that $[A + E \mid b + \beta]$ is Toeplitz.

1. Choose a large number ω
   $E := 0$, $\alpha := 0$, $\beta_1 := 0$, compute x from $min\|Ax - b\|_p$, X from x, $\hat{r} := b - Ax$.
2. **repeat**
   (a)
   $$\underset{\Delta\alpha,\Delta\beta_1,\Delta x}{\text{minimize}} \|M \begin{pmatrix} \Delta\alpha \\ \Delta\beta_1 \\ \Delta x \end{pmatrix} + \begin{pmatrix} -\omega\hat{r}(\alpha,\beta_1,x) \\ D\alpha \\ \beta_1 \end{pmatrix} \|_p$$
   where
   $$M = \begin{pmatrix} \omega(X-\hat{F}) & -\omega e_1 & \omega(A+E) \\ D & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
   (b) $x := x + \Delta x$, $\alpha := \alpha + \Delta\alpha$, $\beta := \alpha$, $\beta_1 := \beta_1 + \Delta\beta_1$.
   (c) Construct E from α, and X from x. $\hat{r} := (b + \beta) - (A + E)x$.

   until ($\|\Delta x\|, \|\Delta\alpha\|, \|\Delta\beta_1\| \leq \epsilon$)

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it is always possible to find a Toeplitz perturbation $[E \mid F\beta]$ such that

$$b + F\beta \in Range(A + E).$$

Therefore, we can expect $\hat{r}$ to become zero when the solution is obtained and since $F\beta$ can play the role of the residual vector, $r = \hat{r} - F\beta = b - (A+E)x$, we can formulate the problem as the following weighted least squares problem

$$\min_{\alpha,\beta,x} \| \begin{pmatrix} \omega\hat{r}(\alpha,\beta,x) \\ D_1\alpha \\ D_2\beta \end{pmatrix} \|_p \qquad (4)$$

where ω is a large number [1, 2] and the diagonal matrices D_1 and D_2 account for the repetition of the elements of α and β in E and $F\beta$, respectively. The minimization required by (4) is done by using a linear approximation to $\hat{r}(\alpha,\beta,x)$. Let Δx and $\Delta\beta$ represent a small change in x and β respectively, and ΔE represent a small change in the variable elements of E. Since we can define a matrix $X \in \mathbf{R}^{m\times q}$ that consists of only 1's and 0's and satisfies $X\alpha = Ex$ [7], we

have $X\Delta\alpha = (\Delta E)x$, where $\Delta\alpha$ represents the corresponding small changes in the elements of α. Then, neglecting the second order terms in $\|\Delta\alpha\|$ and $\|\Delta x\|$, $\hat{r}(\alpha+\Delta\alpha, \beta+\Delta\beta, x+\Delta x) =$ $\hat{r}(\alpha, \beta, x) - X\Delta\alpha + F\Delta\beta - (A+E)\Delta x$.
The linearization of (4) now becomes:

$$\underset{\Delta x, \Delta\alpha, \Delta\beta}{\text{minimize}} \left\| G \begin{pmatrix} \Delta\alpha \\ \Delta\beta \\ \Delta x \end{pmatrix} + \begin{pmatrix} -\omega\hat{r} \\ D_1\alpha \\ D_2\beta \end{pmatrix} \right\|_p$$

where $G = \begin{bmatrix} \omega X & -\omega F & \omega(A+E) \\ D_1 & 0 & 0 \\ 0 & D_2 & 0 \end{bmatrix}$. Now, we discuss backward prediction (similar results hold for forward prediction as well) where we need to impose the Toeplitz structure on $[A \mid b]$. Note that the perturbation in b can be represented using the perturbation in A except for its first component. Specifically, if all the elements on the different diagonals of $[A|b]$ are different and subject to error and $\alpha = (\alpha_1 \cdots \alpha_{n+m-1})^T$, E is Toeplitz with first row $= (\alpha_n \cdots \alpha_1)$ and first column $= (\alpha_n \cdots \alpha_{n+m-1})^T$, $F = I$, $\beta = (\beta_1 \cdots \beta_m)^T$, then since $\beta_i = \alpha_{i-1}$, $i = 2, \cdots, m$, we have

$$\beta = \beta_1 e_1 + \hat{F}\alpha \tag{5}$$

where $e_1 = (1 \quad 0 \quad \cdots \quad 0)^T$ and $\hat{F} = \begin{pmatrix} 0_{1\times(m-1)} & 0_{1\times n} \\ I_{(m-1)\times(m-1)} & 0_{(m-1)\times n} \end{pmatrix}$. Also, from (5), we have

$$X\Delta\alpha - F\Delta\beta + (A+E)\Delta x = (X-\hat{F})\Delta\alpha + (A+E)\Delta x - \Delta\beta_1 e_1.$$

Algorithm STLN-LPR summarizes the computations for solving a Toeplitz structure preserving linear prediction problem where the $(m+n-1)\times(m+n-1)$ matrix D is

$$D^2 = diag(2, 3, \cdots, n+1, n+1, \cdots, 3, 2, 1)$$

provided all $m+n-1$ elements of A are different and subject to error.

3. COMPUTATIONAL RESULTS

The accuracy of Algorithm STLN-LPR is compared with that of the LS and TLS methods in several Toeplitz problems using MATLAB [6]. First, we consider a linear prediction (LPR) problem. We start from $m + n - 1$ data samples $z_i \in \mathbf{C}$ arranged in a Toeplitz matrix $A_c \in \mathbf{C}^{m\times n}$ and set up the zero residual set of equations $A_c x_c = b_c$ by choosing an exact right-hand side vector b_c. Specifically, $m = 42$, $n = 8$, $z_t = \sum_{k=1}^{8} e^{(-d_k + 2\pi\sqrt{-1}f_k)t}$, $1 \le t \le 50$, the first column of A_c is $(z_{42} \cdots z_2\ z_1)^T$ and $b_c = (z_{50} \cdots z_9)^T$, so $[A_c \mid b_c]$ is Toeplitz. The 8 pairs of exact (d_k, f_k) are $(0.1, 0.5), (0.2, 0.4), (0.3, 0.3), (0.35, 0.1), (0.4, 0.2), (0.5, 0.45), (0.05, 0.25), (0.45, 0.05)$. Random errors, normally distributed with mean zero and variance σ_ν^2, are then added to the data z_i and b_c, generating the perturbed set $Ax \approx b$. The relative errors in the solution vector x, $\frac{\|x_\alpha - x_c\|}{\|x_c\|}$, with $\alpha = LS$, TLS or $STLN2$ are averaged over 100 runs per σ_ν. The L_2 norm STLN solution is computed in 10 iterations and the matrix M always has full column rank.

In addition, we show the relative accuracy of damping factor and frequency estimates obtained by rooting the computed LPR polynomial in each run, as in Prony's method. The results are presented in Tables 1-3. Observe that STLN2 improves the accuracy of the TLS estimates with a factor 30 to 40 for noise standard deviations up to 10^{-5}, while the differences in accuracy between TLS and LS remain negligible. For $\sigma_\nu = 10^{-4}$, the frequency accuracy improves even more than 400 times. Algorithm STLN-LPR can also be used for computing the low rank Hankel norm approximation of any

σ_ν	LS	TLS	STLN2
e-10	7.12e-8	7.12e-8	2.04e-9
e-9	6.61e-5	6.61e-5	1.87e-6
e-8	7.19e-4	7.19e-4	1.98e-5
e-7	7.12e-3	7.12e-3	1.97e-4
e-6	6.95e-2	6.94e-2	1.91e-3
e-5	8.03e-1	6.53e-1	2.12e-2
e-4	3.66e+1	8.29e+0	1.86e-1

Table 1: Relative accuracy of the LP solution

σ_ν	LS	TLS	STLN2
e-10	1.34e-7	1.34e-7	4.02e-9
e-9	6.61e-5	6.61e-5	1.87e-6
e-8	7.19e-4	7.19e-4	1.98e-5
e-7	7.12e-3	7.12e-3	1.97e-4
e-6	6.95e-2	6.94e-2	1.91e-3
e-5	8.03e-1	6.53e-1	2.120e-2
e-4	3.66e+1	8.29e+0	1.86e-1

Table 2: Relative accuracy of the damping factor

σ_ν	LS	TLS	STLN2
e-10	1.98e-8	1.98e-8	6.8e-10
e-9	1.98e-5	1.98e-5	6.06e-7
e-8	2.05e-4	2.05e-4	6.40e-6
e-7	1.99e-3	1.99e-3	6.87e-5
e-6	2.00e-2	2.01e-2	6.63e-4
e-5	2.32e-1	1.95e-1	6.94e-3
e-4	6.19e+1	3.06e+1	6.54e-2

Table 3: Relative accuracy of the frequency

	LS	TLS	STLN2	STLN1	STLN∞
b_{err}	8.9e-3	5.3e-4	4.3e-4	5.5e-4	5.7e-4
A_{err}	1.2e-2	1.2e-2	5.3e-4	5.4e-4	5.2e-4
x_{err}	1.5e-2	1.5e-2	7.3e-4	3.5e-4	5.9e-4

Table 4: Solution Accuracy, A Toeplitz, b unstructured, $m = 11$, $n = 6$, $q = 4$

$m \times n$ Hankel matrix H, which is obtained by solving $Ax \approx b$, where A consists of the first $n-1$ columns of H and b is the last column of H. Then the STLN solution yields the minimum perturbation $[E \mid \beta]$ that makes $[A+E \mid b+\beta]$ rank deficient, keeping the Hankel structure.

In the next problems we tested, the matrix A had a Toeplitz structure, but b was unstructured. Random errors were introduced in the elements of A (preserving the Toeplitz structure) and in the elements of b. The relative values of these errors are given by the quantities A_{err} and b_{err}. The values of x were then computed by LS, TLS, and STLN. The structure of A was preserved by STLN, but not by TLS. The relative error in x is denoted by x_{err}. Approximately 100 different problems of this type were solved with $m \geq n+q$ and $m \leq 20$. Table 4 shows typical results for problems with $m = 11$, $n = 6$, and $q = 4$. Results from different size problems were similar. It is seen in Table 4 that the STLN algorithm gives an x vector that is between 20 and 43 times more accurate than that computed by TLS. The accuracy of TLS and LS were very close.

It is known that for some applications, the use of the L_1 norm is more robust than the L_2 norm [4, 5]. In order to explore this with respect to STLN, we carried out a computational test using LS, TLS, and STLN with $p = 1$ and 2, on the same problems. We present in Table 5 some results for $[A \mid b]$ Toeplitz when there is a large perturbation (outlier) in one of the diagonals, and $m = 14$, $n = 4$, and $q = 17$. For each of the six problem listed, a different single diagonal of $[A \mid b]$ was perturbed by 0.5. All other diagonals were randomly perturbed by δ_i, where $|\delta_i| \leq 1.0e-4$. The first element of b was not perturbed (therefore, not corrected). The unperturbed matrix A is given by $A =$ Toeplitz(col, row) where $col = (-2, 0, 10, 11, -1, -2, 20, 32, 9, -5, 38, 84, 50, -1)^T$, row $= (-2, 3, 5, 0)$, and $b_1 = 0$. The exact solution is $x = (1, -1, 1, -1)^T$.

The STLN1 result is essentially unaffected by the outlier. This is in contrast to the LS, TLS and STLN2 results, where the outlier causes x_{err} to be several orders of magnitude larger. For two or three diagonals with large perturbations (0.5), the STLN1 errors were similar to those in Table 5, while the other errors were two to three times larger. However, with four or more

Problem	LS	TLS	STLN1	STLN2
1	9.0e-2	3.0e-1	8.1e-6	9.7e-2
2	2.1e-1	6.5e-2	7.5e-6	7.1e-2
3	2.3e-1	8.9e-3	9.3e-6	2.2e-2
4	2.1e-1	1.5e-2	1.3e-5	4.3e-2
5	2.2e-1	3.9e-2	8.1e-6	4.3e-2
6	1.4e-1	2.4e-2	7.2e-6	3.3e-3

Table 5: x_{err} for $[A \mid b]$ Toeplitz with an outlier

diagonals perturbed by 0.5, all values of x_{err} were large, including STLN1.

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