THE WAVELET TRANSFORM OF HIGHER DIMENSION AND THE RADON TRANSFORM

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ABSTRACT

In this paper, we present a fast algorithm for the computation of the wavelet transform in higher dimensional Euclidean space R^n with arbitrary shaped wavelets. The algorithm is a direct consequence of the convolution property of the Radon transform and shows significant improvement in speed. We also present a novel approach for the computation of the Daubechies type wavelet transform under the Radon transform domain where the n-dimensional multiresolution Analysis (MRA) is reduced to one-dimensional MRA. We found applications of this approach on, for instance, multiresolution reconstruction of the tomographic image with the standard methods of denoising, where determination of wavelet coefficients is required under the Radon transform domain. Along with the possibility of reducing samples angularly with decreasing resolution, the efficiency can be further improved. Besides, extra property such as "rotated" wavelet can be easily implemented with this algorithm.

1. INTRODUCTION

In recent years, the wavelet transform raises much interest in various fields, particularly in the signal processing society. Many applications of the Wavelet transform, such as in the joint time-frequency analysis [1, 2] and in the design of the numerical computational algorithms [3], have demonstrated its performance to often be better than the classical methods. We refer the readers to [1] for tutorial of wavelet. Most of the current applications of wavelet analysis are based on one-dimensional wavelet and the higher dimensional cases are simply tensor product of the one-dimensional wavelet. Although this direct extension also maintains fast algorithms such as the row-column approach, it may lose some extra properties of higher dimensional wavelet, such as the rotation parameter which is used in many applications, for instance fractal analysis [4]. Furthermore, the direct extension of the one-dimensional wavelet transform can hardly be used in the evaluation of the non-separable higher dimensional wavelet transform. In this paper, we present a fast algorithm for the computation of the wavelet transform of higher dimension. Instead of only dealing with those wavelet transforms which are direct extension of the one-dimensional ones, the algorithm is equally efficient in computing the wavelet transform with wavelets of any arbitrary shaped, both separable and non-separable. In the non-separable case, the n-dimensional wavelet transform is basically decomposed into a number of one-dimensional ones to achieve a significant speed improvement. The saving becomes even more significant when the support of wavelet decrease, and the saving also increase with the decrease of the Fourier support of the wavelet since the radial samples can be reduced. For the case that an n-dimensional wavelet transform is associated to a particular multiresolution analysis, the original n-dimensional MRA can be replaced by one-dimensional MRA with the proposed approach. It is a direct consequence of the convolution property of the Radon transform.

We organize this paper as follow: In section 2, we demonstrate a simple relationship between wavelet transform in R^n and the Radon transform, then indicate how the Radon transform can lead to a fast computation of wavelet coefficients of dimension greater or equal to two. In section 3, we develop the Multiresolution Analysis under the Radon transform domain. We consider the computation of wavelet in R^n which are generated by tensor product under the Radon transform domain. It had been shown that such wavelets are of unconditional basis for several functional spaces and they are best basis for statistical estimation [5]. Also, it is trivial for applying one-dimensional MRA on radial variable of the projection domain which defines isotropic/directional wavelets [6]. We also found that our method can easily be used to compute the "rotated" Daubechies' wavelet transform as compared to the traditional approach that it is quite complicated to directly adopt MRA in R^n for this purpose. Finally, we present numerical experimental results to justify our assertion.

2. WAVELET COEFFICIENTS UNDER THE PROJECTION DOMAIN

Basically, the evaluation of the wavelet transform coefficients can be considered as to perform convolution with several scales of a particular mother wavelet with the input signal. It is a well known fact that an n-dimensional convolution can be decomposed into a number of one-dimensional convolutions under projection domain[7], Suppose $\hat{f}(w) = \int f(x)e^{-ixw}dx; i^2 = -1$ (throughout this paper, the integration range from $-\infty$ to ∞ if no limit is specified), $\check{g}$ and $\check{h}$ is the Radon transform of $g, h \in R^n$ respectively, then,

This work is supported by HKP research grant 340/808

0-7803-2431-5/95 $4.00 © 1995 IEEE

$$R(g \underline{*} h) = \check{h} *_\rho \check{g} \quad (1)$$

We refer the readers to [7] for the proof of eqn.1. For the mother wavelet under linear transformation including scaling and rotation,

$$\check{\psi}(A\underline{x})(\rho, \underline{\xi}) = |B| \check{\psi}(\rho, B^T \underline{\xi}) \quad (2)$$

where $B = A^{-1}$. Confine the Linear transform matrix A into isotropic scaling and rotation matrix, we get a much more simpler form,

$$\check{\psi}(A\underline{x})(\rho, \underline{\xi}) = \frac{1}{a^{n-1}} \check{\psi}(a\rho, \underline{\xi} - \underline{\zeta}) \quad (3)$$

where a is dilation factor and $\underline{\zeta}$ is rotation angle. In general, we have the formula for the computation of the wavelet coefficients under the projection domain as follows,

$$R\left[f(\underline{x}) \underline{*} \psi(A\underline{x})\right](\rho, \underline{\xi}) = \check{f}(\rho, \underline{\xi}) *_\rho \frac{1}{a^{n-1}} \check{\psi}(a\rho, \underline{\xi} - \underline{\zeta}) \quad (4)$$

It should be noted that the projected wavelet are not necessarily isotropic, that is, it can be angularly varied. In the context of reconstruction from projection, we have projection data only. We can apply the last equation to the projected data $\check{f}$ and take inversion to the result to obtain the wavelet transform of f.

3. MULTIRESOLUTION ANALYSIS

On the other hand, for the cases that the wavelet transform is associated to a particular multiresolution analysis defined in R^n, we can also compute the wavelet coefficients directly under the Radon transform domain using one-dimensional MRA only. Fourier slice theorem tells us that the radial line spectrum at angle θ of the Fourier transform of a given n-dimensional function is the spectrum of the Radon transform at angle θ. That is,

$$\tilde{\check{f}}(s, \xi) = \int \check{f}(\rho, \underline{\xi}) e^{-i\rho s} d\rho = \int f(\underline{x}) e^{-is\underline{\xi} \cdot \underline{x}} d\underline{x} \quad (5)$$

We will use the Fourier slice theorem to obtain the wavelet coefficients under Radon domain. Consider the wavelet of Daubechies ' type, define $\phi(t)$ and $\psi(t)$ as scaling function and wavelet function respectively first,

$$\phi(t) = \sum_{n=0}^{L-1} h(n)\phi(2t-n); \psi(t) = \sum_{n=0}^{L-1} g(n)\phi(2t-n).$$

with $g(n) = (-1)^n h(L-1-n)$. The coefficient $h(n)$ characterizes the wavelet. The wavelet basis induces "MRA" on $L^2(R)$ so that

$$... \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} ...$$

$$s.t. \bigcap_{j \in Z} V_j = \{\oslash\}, \overline{\bigcup_{j \in Z} V_j} = L^2(R)$$

The number of vanishing moments is related to the filter coefficients by L=2M. This implies $\int \psi(x) x^m dx; m = 0..M-1$. Certainly, this condition can be relaxed for particular application. One advantage of a "multiresolution" architecture is that it leads to an efficient tree-structured algorithm for the computation of the wavelet coefficients. By evaluating $c_n^J = < f, \phi_{Jn} >$ for the resolution J (projecting the function f on V_J), we can obtain the wavelet and scaling coefficients as follows,

$$c_k^j = < f, \phi_{Jk} > = 1/\sqrt{2} \sum_{n=0}^{L-1} h(n-2k) c_n^{j-1}$$

$$d_k^j = < f, \psi_{Jk} > = 1/\sqrt{2} \sum_{n=0}^{L-1} g(n-2k) c_n^{j-1}$$

This recursive style enables us to compute the inner product between wavelets and the function to be analyzed in a discrete fashion. In the Fourier transform domain, the two-scale difference equation can be visualized as,

$$\tilde{\phi}(w) = m_0(w_x/2)\tilde{\phi}(w/2); \tilde{\psi}(w) = m_1(w_y/2)\tilde{\phi}(w/2)$$

where

$$m_0(w) = \sum_{n=0}^{L-1} h(n) e^{-iwn}; m_1(w) = \sum_{n=0}^{L-1} g(n) e^{-iwn} \quad (6)$$

For the two-dimensional MRA,

$$\tilde{\phi}(w_x)\tilde{\phi}(w_y) = m_0(w_x/2)m_0(w_y/2)\tilde{\phi}(w_x/2)\tilde{\phi}(w_y/2)$$

$$\tilde{\phi}(w_x)\tilde{\psi}(w_y) = m_0(w_x/2)m_1(w_y/2)\tilde{\phi}(w_x/2)\tilde{\phi}(w_y/2)$$

$$\tilde{\psi}(w_x)\tilde{\phi}(w_y) = m_1(w_x/2)m_0(w_y/2)\tilde{\phi}(w_x/2)\tilde{\phi}(w_y/2)$$

$$\tilde{\psi}(w_x)\tilde{\psi}(w_y) = m_1(w_x/2)m_1(w_y/2)\tilde{\phi}(w_x/2)\tilde{\phi}(w_y/2)$$

By defining the following:

$$p_0^\theta(w_r) = m_0(w_r cos\theta) m_0(w_r sin\theta) \quad (7a)$$

$$p_1^\theta(w_r) = m_0(w_r cos\theta) m_1(w_r sin\theta) \quad (7b)$$

$$p_2^\theta(w_r) = m_1(w_r cos\theta) m_0(w_r sin\theta) \quad (7c)$$

$$p_3^\theta(w_r) = m_1(w_r cos\theta) m_1(w_r sin\theta) \quad (7d)$$

such that

$$\tilde{\check{\phi}}^\theta(w_r) = p_0^\theta(w_r/2)\tilde{\check{\phi}}^\theta(w_r/2); \tilde{\check{\psi}}_k^\theta(w_r) = p_k^\theta(w_r/2)\tilde{\check{\phi}}^\theta(w_r/2)$$

where $k = 1, 2, 3$. The MRA under the projection domain is then obtained. We can use eqn.6 to compute p_k^θ as given by eqn.7 and use any Inverse Fast Fourier transform algorithm to compute the "projected filter coefficients" such that we can rewrite the above equations as follows,

$$\phi^\theta(r) = \sum_{n=0}^{L(\theta)-1} P_0^\theta(n)\phi^\theta(2r-n);$$

$$\psi_k^\theta(r) = \sum_{n=0}^{L(\theta)-1} P_k^\theta(n)\phi^\theta(2r-n). \quad (8)$$

where $p_k^\theta(w_r) = \sum_{n=0}^{L(\theta)-1} P_k^\theta(n) e^{-inw_r}; k = 0, 1, 2, 3$ Note that if the wavelet to be computed does not have compact Fourier support, other method should be used for the projection of the wavelets provided that the wavelet itself have compact support. But most important, the ramped projected wavelet remains compact and possesses the same number of vanishing moments (see[8]).

As can be seen, eqn.8 is an "Angularly dependent MRA" under the projection domain. At the first glance, the formulation is natural and have no problem. However, let us take a closer look at the spectral support of a general Radon transformed function. Suppose the function have spectral support of radius Ω (inside circle C_2), then 2Ω sampling rate could be enough to sample it. However, it is impossible to represent a MRA under projection domain if we use the same sampling rate to generate the projected wavelet filters. The reason is that the wavelet filters will be aliased and make the ramped wavelet filter not remains localized. Although the scaling filter and lower scale wavelet filters can be represented without aliasing, the aliased wavelet filter of finest scale destroys the "nested architecture". One solution is to compute the wavelet coefficients of level J at level J-1, the result is shown in figs. 6 and 7. Another method is to change the sampling rate of sampled signal, (i.e. change the radial bandwidth from C_2 to C_1 shown in Figure 1). Then, the wavelet filters and scaling filter can be represented in the new sampling rate and $\check{f}(\rho, \theta) \in \overline{span\{\check{\phi}^{\theta}\}}$. The number of samples to be increased is about $1/\sqrt{N}$. It should be noted that the resulting wavelet coefficients should be the same.

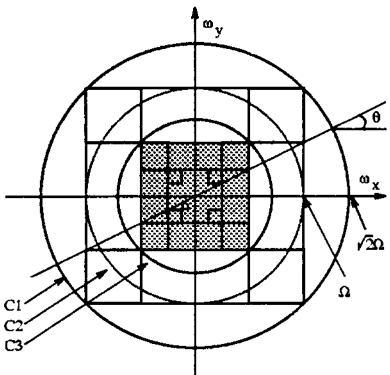

Figure 1: Spectral supports of a function $f \in L^2(R^n)$ and wave-lets.

4. NUMERICAL EXAMPLE

We use a simple numerical example to verify the algorithm for the computation of non-separable two-dimensional wavelet transform. Firstly, we define a wavelet R^2, for example the Mexican hat $(1 - x^2 - y^2)e^{-(\frac{x^2+y^2}{2})}$. Then we perform a two-dimensional convolution with a sample two-dimensional image and obtain a standard wavelet transformed signal. Secondly, we perform the fast algorithm we proposed: we obtain the projections of the wavelet and the sample image and then perform an one-dimensional convolution between them radially. Then, we obtain a Radon transformed wavelet coefficient for a fixed scale and rotation. The result is then filter-backprojected to obtain the required wavelet coefficients. A comparison of the standard approach and the proposed approach is shown in fig.2. It is seen that they are similar. Also, the Mexican hat and the projected Mexican hat is shown in fig.3.

We realized the proposed algorithm on a SUN Sparc 10 workstation and made use of the system call *getrusage()* to analysis the time used by the operations, not including the system usage. We took five trials on each scale and used the average to compare the performance. It is summarized in table 1. The analysis is rough since we do not address the sampling requirement but it already shows the significant improvement in performance.

On the other hand, we realized the Multiresolution Analysis under Radon transform domain by using Daubechies' filter of 2 vanishing moment (four coefficients). The wavelet coefficients and scaling coefficients using our method are shown in figs.4-7. In our simulation, the function are projected in 256 angles, each with 256 samples. For figs.4 and 5, the finest scale scaling filter is adjusted to $1/\sqrt{2}$ bandwidth of the sampled signal whereas for figure 4 and 5, the finest wavelet filter are obtained at the same sampling rate of the sampled projection data so that the finest scale scaling filter is half the bandwidth of the sampled projection data.

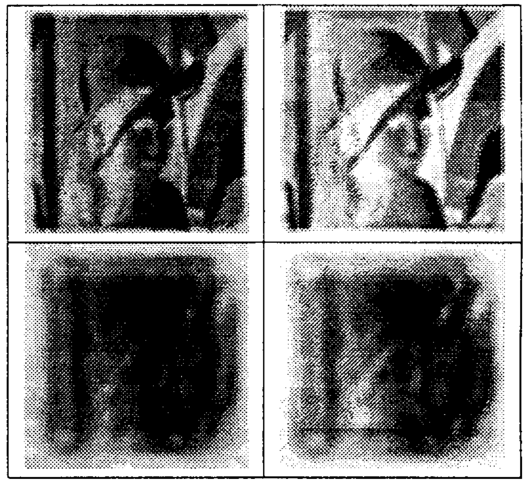

Figure 2: Up-Left is Wavelet transform of Lenna (256x256) with Mexican (8x8), Up-Right is the corresponding projection domain result. Bottom-Left is Wavelet transform of Lenna (256x256) with Mexican (64x64), Bottom-Right is the corresponding projection domain result.

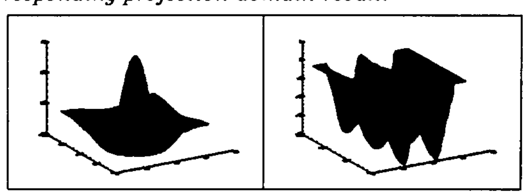

Figure 3: Left is Mexican, Right projected Mexican hat.

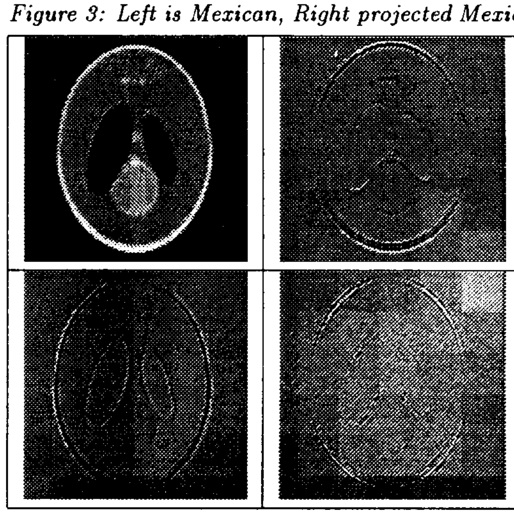

Figure 4: MRA in Radon domain of scale 1 with bandwidth of the finest scale Scaling filter equal to $\sqrt{2}$ of projection data's.

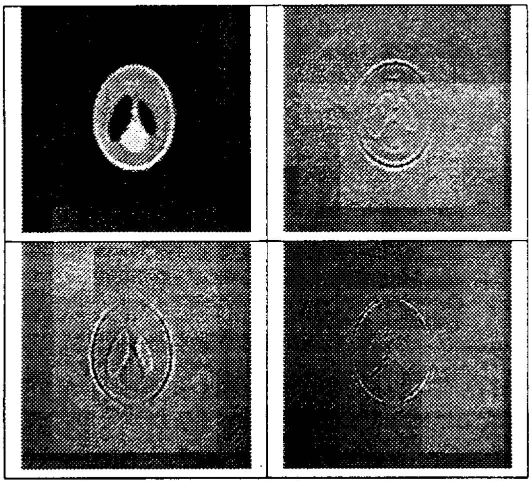

Figure 5: MRA in Radon domain of scale 2 with bandwidth of the finest scale Scaling filter equal to $\sqrt{2}$ of projection data's.

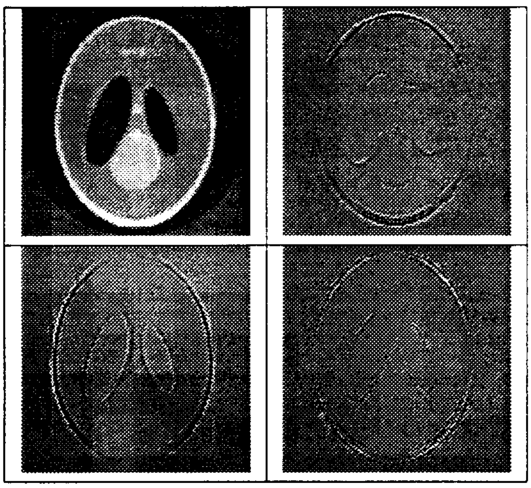

Figure 6: MRA in Radon domain of scale 1 with bandwidth of the finest scale Scaling filter equal to half of projection data's.

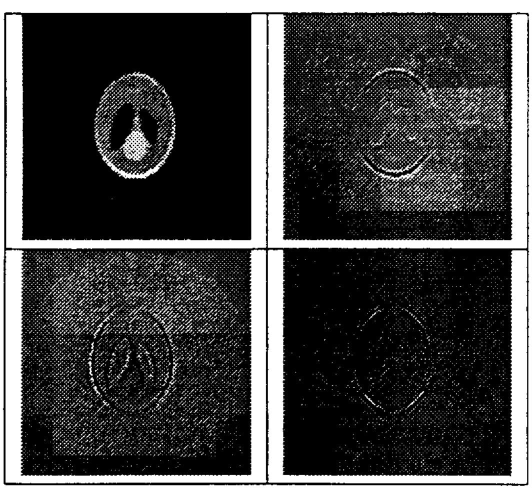

Figure 7: MRA in Radon domain of scale 2 with bandwidth of the finest scale Scaling filter equal to half of projection data's.

5. CONCLUSION

In this paper, we present a fast computational algorithm for higher dimensional wavelet transform of arbitrary shaped wavelet by means of Radon transform. We consider the computation of the separable and non-separable n-dimensional wavelet transform. In non-separable case, the standard n-dimensional convolution approach is basically reduced to some one-dimensional convolutions. The saving becomes significant when the support of wavelet decrease, and the saving also increases with the decrease of the bandwidth of wavelet since the radial samples can be reduced. For the wavelet transform which is associated to a particular n-dimensional multiresolution analysis, the use of the proposed approach reduces the n-dimensional MRA to become one-dimensional MRA. We found applications of this MRA construction on the reconstruction of tomographic image.

Size of wavelet	2-dimensional Convolution(sec)	Projection method(sec)
8x8	45.48	28.51
16x16	100.13	29.00
32x32	248.51	30.17
64x64	719.99	32.85
128x128	2310.77	39.96

Table 1: Timing of convolution and projection method for wave-let coefficient calculation.

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