Wavelet Interpolation: From Orthonormal to the Oversampled Wavelet Transform

Nurgun Erdol, Feng Bao and Zajing Chen
Department of Electrical Engineering
Florida Atlantic University
Boca Raton, Fl 33431

Abstract

The orthonormal wavelet transform is an efficient way for signal representation since there is no redundancy in its expression, but due to aliasing in the decimation stage it lacks the often desired property of shift invariance. On the other hand, the oversampled or nonorthogonal wavelet offers a finer resolution in translation; thus reducing the effect of shift of origin, it becomes more robust to changes in the initial phase of the signal. In some areas of signal processing, such as wideband correlation processing, sensitivity to time alignment necessitates the use of the nonorthogonal wavelet transform. The price paid for the advantage of robustness to shifting is the introduction of redundancy in the expression. In many applications, both of these two properties are needed in different stages of signal processing. Thus there is a need to know the conditions under which the redundant and nonorthonormal wavelet transform coefficients can be derived from the orthonormal wavelet transform coefficients. The answer provides us with a convenient way to switch between these two forms: the orthonormal wavelet for efficient expression, and the nonorthogonal one whenever it is necessary for feature extraction.

1. Introduction

The orthonormal wavelet transform is an efficient way for signal representation since there is no redundancy in its expression[1, 2], but due to aliasing in the decimation stage it lacks the often desired property of shift invariance. On the other hand, the oversampled and nonorthogonal wavelet transform offers finer resolution in translation; thus reducing the effect of shift of the time of origin it becomes more robust to changes in the initial phase of the signal[3, 4, 5, 6]. The discrete wavelet transform with no decimation at all is shift invariant. In some areas of signal processing, such as wide band correlation processing, sensitivity to time alignment necessitates using the non-orthogonal wavelet transform[7, 8, 9, 10] . For increased reliability in detection, scale parameters between powers of 2 are also needed to obtain wavelets that are more closely matched to the signal. The price paid for the advantage of robustness to shifting as well as scale matching is the introduction of redundancy in the transformation. In many applications, both of these two properties are needed in different stages of signal processing. For example, data acquisition, transmission and storage aim for efficiency, but when processing signals for detection, it is more important to get an accurate scale and time shift. Thus there is a need to know the conditions under which the redundant and nonorthonormal wavelet transform coefficients can be derived from the orthonormal wavelet transform coefficients. The answer provides us with a convenient way to switch between these two forms: the orthonormal wavelet for efficient expression, and the nonorthogonal one whenever it is necessary for feature extraction.

2. Interpolation Matrix

Because the orthonormal wavelet transform contains complete information about a signal, its conversion to an oversampled one is a problem of interpolation. We consider interpolation along the constant-scale translation axis as well as constant-translation scale axis.

For convenience and without loss of generality, let the finest resolution of the signal $x(t)$ be on scale P. We have

$$x(t) = 2^{P/2} \sum_n c_n^P \phi(2^P t - n) \tag{2.1}$$

where $\phi(t)$ is the scaling function and c_n^P are the scaling coefficients at scale P. The signal $x(t)$ can also be expressed as

$$\begin{aligned} x(t) = &\sum_n \sum_{m=0}^{P-1} 2^{m/2} d_n^m \psi(2^m t - n) \\ &+ \sum_n c_n^0 \phi(t-n) \end{aligned} \tag{2.2}$$

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where $\psi(t)$ is the mother wavelet function and d_n^m are the wavelet coefficients at scale m. The scaling and wavelet coefficients are calculated as the inner products

$$\begin{cases} c_n^m = 2^{m/2} \int_{-\infty}^{+\infty} x(t) \cdot \phi(2^m t - n) dt \\ d_n^m = 2^{m/2} \int_{-\infty}^{+\infty} x(t) \cdot \psi(2^m t - n) dt. \end{cases} \quad (2.3)$$

The wavelet coefficients at scale m, $m \in \{0, 1, 2, \cdots, P\}$, and scaling coefficients at scale 0 produce an exact expression for the signal.

2.1. Interpolation along the translation axis

The oversampled wavelet coefficients at scale 0 with an oversampling rate of 2^P are given by

$$\begin{cases} \overset{\sim 0}{d}_{j \cdot 2^P + k} = \int_{-\infty}^{+\infty} x(t) \cdot \psi(t - j - k/2^P) dt \\ k \in \{0, 1, 2, \cdots, 2^P - 1\} \end{cases} \quad (2.4)$$

Substituting eq.(2) in (4), yields

$$\overset{\sim 0}{d}_{j \cdot 2^P + k} = \sum_n \sum_{m=0}^{P-1} 2^{m/2} d_n^m \cdot \left\{ \int_{-\infty}^{+\infty} \psi(2^m t - (n - 2^m \cdot j)) \cdot \psi(t - k/2^P) dt + \right\} + \sum_n c_n^0 \cdot \int_{-\infty}^{+\infty} \phi(t - (n - j)) \cdot \psi(t - k/2^P) dt \quad (2.5)$$

Let:

$$\begin{cases} \xi_{n,k}^m = 2^{m/2} \int_{-\infty}^{+\infty} \psi(2^m t - n) \cdot \psi(t - k/2^P) dt \\ \eta_{n,k} = \int_{-\infty}^{+\infty} \phi(t - n) \cdot \psi(t - k/2^P) dt \end{cases} \quad (2.6)$$

For compactly supported wavelets and scaling functions, $\xi_{n,k}^m$ and $\eta_{n,k}$ are zero for many values of n. For example, if $\psi(t)$ has unit support, then $\xi_{n,k}^m$ vanish except for integer values of n in $[2^m(k/2^P - 2^{-m}), 2^m(k/2^P + 1)]$ or for n in the interval with sufficient bounds of $[-1, 2^{P+1} - 1]$. Then, eq.(2.5) can be rewritten as

$$\overset{\sim 0}{d}_{j \cdot 2^P + k} = \sum_{m=0}^{P-1} \sum_{n=n_0}^{n_1} d_n^m \cdot \xi_{n-2^m \cdot j, k}^m + \sum_{n=n_2}^{n_3} c_n^0 \cdot \eta_{n-j,k} \quad (2.7)$$

One way to implement eq. (2.7) is to consider P sequences $\left\{\xi_{n,k}^m\right\}_{m=0}^{P-1}$ and realize the n-sum of the first term as a convolution, respectively, with P sequences $\{d_{-n}^m\}_{m=0}^{P-1}$. Each convolution is subsampled by $2^m \cdot j$ and summed over m. The process has to be repeated for each j and k. A more elegant and efficient formulation is to reconsider eq. (2.6) and define the related cross correlation function

$$\widetilde{r_m}(\tau_1, \tau_2) = 2^{m/2} \int_{-\infty}^{+\infty} \psi(2^m t - \tau_1) \cdot \psi(t - \tau_2) dt. \quad (2.8)$$

Sampling r at $\tau_1 = 2^{m-P} l$ and $\tau_2 = 2^{-P} k$ yields

$$\begin{aligned} r_m(l, k) &= 2^{m/2} \int_{-\infty}^{+\infty} \psi(2^m t - 2^{m-P} l) \cdot \psi(t - 2^{-P} k) dt. \\ &= 2^{m/2} \int_{-\infty}^{+\infty} \psi(2^m t - 2^{m-P}(l - k)) \cdot \psi(t) dt \\ &= r_m(l - k) \end{aligned}$$

It is clear that $\xi_{n-2^m \cdot j, k}^m = r_m(2^{P-m} n - 2^P j - k)$ and eq. (7) contains the convolution of the sequence d_n^m with the sequence r_m which has been decimated by the factor 2^{P-m}.

$$\overset{\sim 0}{d}_{j \cdot 2^P + k} = \sum_{m=0}^{P-1} \sum_{n=n_0}^{n_1} d_n^m \cdot r_m(2^{P-m} n - 2^P j - k) + \sum_{n=n_2}^{n_3} c_n^0 \cdot \eta_{n-j,k} \quad (2.9)$$

Alternatively, define the sequence $\delta_n^m = d_l^m$ when $n = l 2^{P-m}$ where l is an integer and $\delta_n^m = 0$ otherwise. We have

$$\overset{\sim 0}{d}_{j \cdot 2^P + k} = \sum_{m=0}^{P-1} \sum_{n=0}^{N_P} \delta_n^m \cdot r_m(n - 2^P j - k) + \sum_n c_n^0 \cdot \eta_{n-j,k} \quad (2.10)$$

where $N_P = (N - 1) 2^{P-m}$ has been chosen as a limit on n for convenience. Define

$$\tilde{\mathbf{d}}_j^0 = \left[\overset{\sim 0}{d}_{j \cdot 2^P}, \overset{\sim 0}{d}_{j \cdot 2^P + 1}, \ldots, \overset{\sim 0}{d}_{j \cdot 2^P + 2^P - 1} \right]^T,$$

$$\mathbf{\Delta}_m = \left[d_0^m, \underbrace{0, .., 0}_{2^{P-m} \text{ zeroes}}, d_1^m, 0, .., 0, d_{N-1}^m \right]^T$$

and

$$\mathbf{\Gamma} = \left[\gamma_0^m = c_0^0, 0, .., \gamma_{2^P}^0 = c_1^0, 0, .., \gamma_{(N-1)2^P}^0 = c_{N-1}^0\right]^T.$$

Then

$$\tilde{\mathrm{d}}_j^0 = \sum_{m=0}^{P-1} \mathbf{R}_m \mathbf{\Delta}_m + \mathbf{\Xi}\mathbf{\Gamma} \tag{2.11}$$

where the interpolation matrices $\mathbf{R}_m$ and $\mathbf{\Xi}$ are given respectively by

$$\mathbf{R}_m = \left[\begin{array}{ccccc} r_m(\alpha) & r_m(\alpha+1) & \cdots & & r_m(\beta) \\ r_m(\alpha-1) & r_m(\alpha) & r_m(\alpha+1) & \cdot & r_m(\beta-1) \\ \cdots & \cdots & \cdots & \cdot & \cdots \\ \cdots & \cdots & \cdots & \cdot & \cdots \\ r_m(\alpha-\gamma) & \cdots & \cdots & \cdot & r_m(\beta-\gamma) \end{array}\right], \tag{2.12}$$

$$\mathbf{\Xi} = \left[\begin{array}{ccccc} \varsigma(\alpha) & \varsigma(\alpha+1) & \cdots & \cdot & \varsigma(\beta) \\ \varsigma(\alpha-1) & \varsigma(\alpha) & \varsigma(\alpha+1) & \cdots & \varsigma(\beta-1) \\ \cdots & \cdots & \cdots & & \\ \cdots & & \cdots & & \\ \varsigma(\alpha-\gamma) & \cdots & & 0 & \varsigma(\beta-\gamma) \end{array}\right], \tag{2.13}$$

with $\alpha = -2^P j$ and $\beta = \alpha + (N-1)2^{P-m}$ and $\gamma = 2^{P-1}$. We also have used

$$\varsigma(n) = \int_{-\infty}^{+\infty} \phi(t - 2^{-P}n) \cdot \psi(t)dt.$$

2.2. Interpolation along the scale axis

The wavelet transform coefficients are defined at fractional scales $m + l/L$ for $l \in \{1, 2, .., L-1\}$ and for convenience we limit the integer scale to $m \in \{1, 2, .., P-1\}$.We have

$$d_n^{-mL+l} = 2^{(m+l/L)/2} \int x(t)\psi(2^{m+l/L}t - n)dt \tag{2.14}$$

As in the previous case, we substitute for $x(t)$ from eq.(2.2) and obtain

$$\begin{array}{c} \tilde{d}_n^{m+l/L} = \sum_{k=0}^{N-1} \sum_{j=0}^{P-1} d_k^j \lambda(m-j, l, k, n) \\ + \sum_{k=0}^{N-1} c_k^0 \rho(m, l, n, k) \end{array} \tag{2.15}$$

where limits on the translation variable k have been chosen for convenience and depend on the wavelets used; and where

$$\begin{array}{c} \lambda(m-j, l, k, n) = 2^{(m-j+l/L)/2}. \\ \int \psi(2^{m-j+l/L}t + 2^{m-j+l/L}k - n)\psi(t)dt, \end{array} \tag{2.16}$$

and

$$\begin{array}{c} \rho_n^m(l, k) = 2^{(m+l/L)/2} \\ \rho_n^m(l, k) = 2^{(m+l/L)/2} \\ \int \psi(2^{m+l/L}t + 2^{m+l/L}k - n)\phi(t)dt. \end{array} \tag{2.17}$$

As before, we define

$$\begin{array}{c} \lambda_k(i, l, n) = 2^{(i+l)/2L} \\ \int \psi(2^{(i+l)/L}t + 2^{(i+l)/L}k - n)\psi(t)dt \end{array} \tag{2.18}$$

which is related to λ by $\lambda_k^n(Li, l) = \lambda(i, l, k, n)$. Eq. (2.15) can now be rewritten as

$$\begin{array}{c} \tilde{d}_n^{m+l/L} = \sum_{k=0}^{N-1} \sum_{i=m-P+1}^{m} d_k^{m-i} \lambda_k^n(Li, l) \\ + \sum_{k=0}^{N-1} c_k^0 \rho_n^m(l, k). \end{array} \tag{2.19}$$

Define the vectors

$$\mathbf{D}_k = [d_k^0, \underbrace{0, ..0}_{(L-1)\ zeroes}, d_k^1, 0, ..0, d_k^{P-1}]^T$$

$$\tilde{\mathbf{D}}_n^m = [\tilde{d}_n^{m+1/L}, \tilde{d}_n^{m+2/L}, .., \tilde{d}_n^{m+(L-1)/L}]^T$$

and

$$\mathbf{C}^0 = [c_0^0, c_1^0, ..c_{N-1}^0]^T$$

, with $I = L(m-P+1)$ and $J = (P-1)L+1$, the matrix $\mathbf{\Lambda}_k =$

$$\left[\begin{array}{cccc} \lambda_k^n(I, J) & \lambda_k^n(I, J-1) & \ldots & \lambda_k^n(I, 0) \\ \lambda_k^n(I, J-1) & \lambda_k^n(I, J-2) & \ldots & \lambda_k^n(I, 1) \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_k^n(I, J-L+1) & \ldots & \ldots & \lambda_k^n(I, L-1) \end{array}\right].$$

and the LXN matrix $\mathbf{\Pi}$ with $(\mathbf{\Pi})_{l,k} = \rho_n^m(l, k)$. We thus have

$$\tilde{\mathbf{D}}_n^m = \sum_k \mathbf{\Lambda}_k \mathbf{D}_k + \mathbf{\Pi}\mathbf{C}^0. \tag{2.20}$$

We note that $\mathbf{\Lambda}_k$ is defined by the entries in its first row and column and its operation on $\mathbf{D}_k$ defines a convolution. The same is not true about the matrix $\mathbf{\Pi}$ because of the two kinds of translation variables involved: $2^{m+l/L}k$ and n. The term $2^{m+l/L}k - n$ cannot be written in terms of $k - n$ due to the fact that $2^{1/L}$ is an irrational number for all integers $L > 1$.

It is also possible to follow the same method used here, to derive coefficients at fractional scales and translation points. Although it is straightforward, the derivation is omitted in this paper.

2.3. Interpolation with oversampling rate of other than 2^P

In the derivation of the translation axis interpolation formula, it was assumed that the signal was scale limited to level P and that the oversampling rate was chosen as 2^P. In some applications, that choice may be limiting. When the oversampling rate is 2^Q, $Q > P$, we recall that the signal is scale limited to P, so that

$$d_n^m \equiv 0, \quad m > P$$

Then the only changes are in the interpolation matrix:

$$\begin{cases} \xi_{n,k}^m = 2^{m/2} \int_{-\infty}^{+\infty} \psi(2^m t - n) \cdot \psi(t - k/2^Q) dt \\ \eta_{n,k} = \int_{-\infty}^{+\infty} \phi(t-n) \cdot \psi(t - k/2^Q) dt \\ m = 0, 1, 2, \cdots, P. \end{cases} \tag{2.21}$$

When the sampling rate is lower than 2^P, or $Q < P$, then

$$\xi_{n,k}^m = 2^{m/2} \int_{-\infty}^{+\infty} \psi(2^m t - n) \cdot \psi(t - k/2^Q) dt = 0, \quad for \;\; m > Q. \tag{2.22}$$

due to the orthogonality of the wavelet functions at a scale higher than Q and the scale 0.

When the oversampling rate is not a power of 2, the basic idea still works, although the interpolation matrix may be more difficult to obtain.

3. Implementation with Filter banks

Use of the interpolation matrix provides us an easy way to get the wavelet samples at an arbitrary position of scale and translation. The idea is useful especially when we are only interested in some special positions, e.g. the sample with the largest energy of projection onto scale 0. In hardware implementation, the price for doing so is that the control flow will be of a high complexity, therefore in many application environments, the method may still not be very attractive.

For an easy implementation of oversampling of the translation axis, especially for real time systems, use of filter banks gives us another possible way. It has less flexibility compared to the interpolation matrix, but its hardware structure is much simpler. A direct way to do this is, to first pass all the orthonormal wavelet samples through the synthesis filter banks to get the reconstructed signal, then use it to get the oversampled wavelet coefficients. Combining the synthesis filter bank for orthonormal wavelet transform and analysis filter bank for nonorthonormal wavelet, we get a useful structure.

4. Conclusion

Methods to interpolate wavelet transform coefficients along the translation as well as the scale axes are discussed. Interpolation matrices are derived. Alternatives to implementation of the translation axis oversampling by using filter banks are suggested. Use of interpolation matrices for interpolation in either domain is practically preferable as samples at any rate and any translation point are readily available.

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