GABOR-TYPE MATRICES AND DISCRETE HUGE GABOR TRANSFORMS

Sigang Qiu

Department of Mathematics
University of Connecticut
U-9, Storrs
CT 06269-3009, USA
sigang@math.uconn.edu

Hans. G. Feichtinger

Department of Mathematics
University of Vienna
Strudlhofgasse 4, A-1090
Vienna, Austria
fei@tyche.mat.univie.ac.at

ABSTRACT

We introduce a class of matrices, called **Gabor-type matrices** and show that the product of two Gabor-type matrices is again a Gabor-type matrix of the same type. The key point for applications is based on the observation that the multiplication of Gabor-type matrices can be replaced by some special "multiplication" of associated small block matrices. We propose an efficient algorithm, which we call the **block-multiplication,** and which makes explicit use of the sparsity of those Gabor-type matrices. As an interesting consequence, we show that Gabor operators corresponding to Gabor triples (g_k, a, b) commute for arbitrary signals $g_k (k = 1, 2)$ provided that ab divides the signal length.

1. INTRODUCTION

A Gabor family is obtained from a Gabor atom (or Gabor window, or basic building block) by time-frequency shifts along some discrete TF-lattice. Such a family is usually not orthogonal. Therefore the determination of appropriate coefficients in order to obtain a series representation of a given signal in terms of this family has been considered a computational intensive task for a long time. Meanwhile it is well known, that it is enough to determine some so-called *dual* Gabor atom γ (with respect to the given lattice generated by (a, b)). The samples of the STFT of the given signal, with the (conjugate of the) dual Gabor atom as analysis window, are then an appropriate set of coefficients.

In a number of papers [4, 5, 6, 7, 10, 11, 12], authors have described several practical approaches to calculate the dual Gabor atom for the discrete and periodic (= finite) setting. Many of those methods however are very much restricted in terms of the size of the window, due to limitations of the size of matrices that can be handled or inverted on a given computer.

In order to imitate the continuous Gabor transform, it is natural to sample the Gabor window sufficiently dense and obtain an approximate dual Gabor window by means of discrete methods. Even if the support of the window itself is not large, a high sampling rate will lead to a discrete Gabor transform with "huge" window size.

The aim of this note is to propose a new way of performing the multiplication for a special class of matrices, which will be called Gabor-type matrices. We will substitute ordinary matrix multiplication and also matrix inversion by a special block matrix "multiplication" and a related inversion procedure. Based on this special multiplication, we can calculate the dual Gabor window and the inverse of the Gabor frame matrix for "huge" Gabor windows. For a much more detailed discussion we refer to [9].

Definition 1 (Gabor-type Block Matrix) *We call an $N \times a$ matrix B a Gabor (a, b)-type block matrix if all the nonzero entries of B are distributed in the k-th subdiagonals only: (for $k = 0, \pm\tilde{b}, \pm 2\tilde{b}, \ldots, \pm(b-1)\tilde{b}$ with $\tilde{b} = N/b$). Equivalently, B can be written as*

$$B = \begin{pmatrix} c_{1,1} & 0 & .. & 0 \\ 0 & c_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{a,a} \\ c_{\tilde{b}+1,1} & 0 & \cdots & 0 \\ 0 & c_{\tilde{b}+2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{\tilde{b}+a,a} \\ \vdots & \vdots & \cdots & \vdots \\ c_{(b-1)\tilde{b}+1,1} & 0 & \cdots & 0 \\ 0 & c_{(b-1)\tilde{b}+2,2} & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{(b-1)\tilde{b}+a,a} \end{pmatrix}$$

where $c_{k,l} \in \mathbb{C}$.

We call $b \times a$ matrix $\tilde{B}$ the associated "nonzero"-block matrix. $\tilde{B}$ is constructed from B in the following way:

$$\tilde{B} = \begin{pmatrix} c_{1,1} & c_{2,2} & \cdots & c_{a,a} \\ c_{\tilde{b}+1,1} & c_{\tilde{b}+2,2} & \cdots & c_{\tilde{b}+a,a} \\ \vdots & \vdots & \cdots & \vdots \\ c_{(b-1)\tilde{b}+1,1} & c_{(b-1)\tilde{b}+2,2} & \cdots & c_{(b-1)\tilde{b}+a,a} \end{pmatrix}$$

Besides, we keep the notations introduced in [7]. Specifically,

0-7803-2431-5/95 $4.00 © 1995 IEEE

(1). The *signal* is always viewed as a row vector in $\mathbb{C}^N$ or equivalently as some N-periodic double infinite sequence. a and b denote *lattice constants*, which means they have to be divisors of N. We call $\tilde{a} = \frac{N}{a}$ and $\tilde{b} = \frac{N}{b}$ the associated *dual lattice constants.*

(2). The *rotation operator* with *rotation number* a acts on vectors as follows:

$rot(x, a) := (x_{N-a}, x_{N-a+1}, \ldots, x_{N-1}, x_0, \ldots, x_{N-a-1})$.

Similarly we can define the rotation operator rot on column vectors.

(3). The *matrix rotation* on matrix B with rotation number a is understood as a rotation acting on all the column vectors of B. That is,

$rotm(B, a) := (rot(B_1, a), rot(B_2, a), \ldots, rot(B_m, a))$, where B_l is the l-th column vector of B, $l = 1, 2, \ldots, m$.

Definition 2 (Gabor-type Matrix) *We say that an $N \times N$ matrix G is a Gabor (a, b)-type matrix if G can be written as*

$$G = \left[B,\ rotm(B, a),\ \ldots,\ rotm^{\tilde{a}-1}(B, a)\right]$$

where B is an $N \times a$ Gabor (a, b)-type block matrix and $rotm = rotm(\cdot, a)$ is the matrix rotation operator. We will call B the Gabor-type block matrix associated to G. $\tilde{B}$ is also called the "nonzero"-block matrix associated to G.

Remarks

It is obvious that a Gabor-type matrix G, the associated block matrix B and "nonzero"-block $\tilde{B}$ determine each other completely. Therefore, we only need to work with the small size "nonzero"-block matrix $\tilde{B}$.

Write $G = (a_{k,l})_{N\times N}$, we usually consider the entry $a_{k,l}$ as N-periodic both in column and row subscripts. Equivalently, $a_{i+N,j+N} = a_{i,j}$ for $i, j \in \mathbb{Z}$.

By the definition of the Gabor-type matrix and $rotm$, it is easy to check that the k-th (for $k = 0, \pm\tilde{b}, \pm 2\tilde{b}, \ldots, \pm(b-1)\tilde{b}$) subdiagonals of G are a-periodic. Let $B = (t_{i,j})_{N\times a}$, and $\tilde{B} = (s_{i,j})_{b\times a}$, then we always assume that $t_{i+N,j+a} = t_{i,j}$ and $s_{i+b,j+a} = s_{i,j}$ for $i, j \in \mathbb{Z}$. ■

2. MAIN RESULTS

In this section we present some of our main results and propose an algorithm to manipulate discrete huge Gabor transforms [9].

Theorem 1 *The matrix product of two Gabor (a, b)-type matrices is itself a Gabor (a, b)-type matrix. Furthermore, a non-singular matrix is of Gabor (a, b)-type if and only if its inverse is of the same type.*

Since Gabor-type matrices are completely determined by the associated "nonzero"-block matrices, it is possible to describe matrix multiplication directly in terms of those small block matrices. The following theorem gives the details of this new "block-matrix multiplication".

Theorem 2 (Block-Multiplication) *Under the assumptions of Theorem 1, assume that $\tilde{B}_1 = (u_{k,l})_{b\times a}$ and $\tilde{B}_2 = (v_{k,l})_{b\times a}$ are two Gabor-type "nonzero"-block matrices associated to Gabor (a, b)-type matrices G_1 and G_2. Let $\tilde{B} := (w_{i,j})_{b\times a}$ be the corresponding "nonzero"-block matrix for the matrix product $G = G_1 * G_2$ Then, for $q = 1, 2, \ldots, b$ and $s = 1, 2, \ldots, a$, the general entry of $\tilde{B}$ can be calculated via the formula:*

$$w_{q,s} = \sum_{p=1}^{b} u_{r_1(p,q),r_2(p,s)} v_{p,s},$$

where $r_1(p, q) = mod(b + q - p + 1, b)$ and $r_2(p, s) = mod\left(s + (p-1)\tilde{b}, a\right)$.

Using the concept of matrix algebra [1], we have the following statements.

Theorem 3 *Let*

$$\mathcal{G} = \{G \in \mathbb{C}^{N\times N} : G \text{ is Gabor } (a,\ b)\text{-type}\}.$$

Then $\mathcal{G} \subset \mathbb{C}^{N\times N}$ is matrix algebra of dimension ab. Furthermore, if ab divides N, then $\mathcal{G}$ is an ab-dimensional commutative matrix algebra.

We call $\mathcal{G}$ a **Gabor-type matrix algebra.** It is easy to see that if $G \in \mathcal{G}$ is nonsingular then $G^{-1} \in \mathcal{G}$.

Let $g \in \mathbb{C}^N$ is a Gabor atom and (a, b) be lattice constants (i.e., divisors of N). Using the notation given in [7], $\text{GAB} = \text{GAB}(g, a, b)$ is the $\frac{N^2}{ab} \times N$ Gabor basic matrix whose row vectors form the Gabor family $\{M_{mb}T_{na}g\}$. The Gabor operator S corresponding to (g, a, b) has the following matrix representation:

$$Sx = x * \left(\text{GAB}' * \text{GAB}\right) \quad \text{for } x \in \mathbb{C}^N,$$

where we call $G = \text{GAB}' * \text{GAB}$ the associated Gabor matrix. Since Gabor matrices are Gabor-type matrices [7], we can easily deduce an interesting consequence of Theorem 3.

Corollary 1 *Let (g_k, a, b) be two Gabor triples for $k = 1, 2$, the associated Gabor operators S_k are defined as*

$$S_k x = \sum_{n=0}^{\frac{N}{a}-1} \sum_{m=0}^{\frac{N}{b}-1} \langle x,\ M_{mb}T_{na}g_k \rangle M_{mb}T_{na}g_k$$

where $x \in \mathbb{C}^N$. Assume that ab divides N (especially, the critical sampling case where $ab = N$), then the Gabor operators S_k for $k = 1, 2$ commute , i.e.

$$S_1 S_2 = S_2 S_1.$$

Applying the special block-multiplication established in Theorem 2, we obtain the following algorithm, which is called the **block-frame (BKFR)** method. It is a very efficient method to invert the frame operator and relies on a combination of two ideas: First, the "slim" matrix multiplication, and secondly the idea to replace the standard Neumann series expansion of the inverse frame operator (usually described in terms of a recursion) by the so-called "power-of-two" trick, which is based on the product representation for partial sums of the Neumann series of order $2^l - 1$ as described in [2, 3]. This approach can be used to compute the dual Gabor atom and is workable even for huge Gabor atom. The details are given in [9].

Algorithm 1 (BKFR-method) *Let (g, a, b) be any Gabor triple which generates frame and G be the associated Gabor frame matrix. Then there exists some positive constant $r_0, (0 < r_0 < 1)$ such that for any $r, 0 < r \leq r_0, Q := rG$ satisfies $||I - Q|| \leq 1 - r||G|| := \gamma < 1$ and therefore $Q^{-1} := (rG)^{-1}$ and the dual Gabor atom $\tilde{g}$ can be obtained iteratively. Setting $\tilde{g}_0 = rg$ and for $n \geq 0$,*

$$\tilde{g}_{n+1} = \tilde{g}_n * \left(I + (I - Q)^{2^n}\right), \tag{1}$$

one obtains

$$\lim_{n\to\infty} \tilde{g}_n = \tilde{g}$$

with the error estimate after n iterations

$$||\tilde{g} - \tilde{g}_n|| \leq \gamma^{2^n} ||\tilde{g}||$$

Remarks

- Since G is a Gabor (a, b)-type matrix [7], $I - Q$ is also a Gabor (a, b)-type matrix. $(I - Q)^{2^n}$ for $n = 0, 1, \ldots$ in Eq. (1) can be calculated iteratively via the special $b \times a$ block-matrix multiplication by Theorem 2. Therefore the algorithm 1 can be performed very efficiently.
- The above approach requires ideally the precise knowledge of the frame bounds. Since the (best) Gabor frame bounds are usually not easily determined, we have to use some reasonable upper bound U_b for the frame operator. The following turns out to be quite useful [8]:
$$U_b = ||G||_1 = ||\tilde{B}||_1$$
where $\tilde{B}$ is the "nonzero"-block matrix corresponding to G. $\tilde{B}$ can be computed directly from the triple (g, a, b) [7]. For a matrix $A = (a_{i,j})_{n,m} \in \mathbb{C}^{n\times m}$, the 1-norm of A is defined as:
$||A||_1 = \max_{1\leq j m} \sum_{i=1}^{n} |a_{i,j}|$.
Then we can take
$$r = \frac{2}{U_b + \frac{1}{U_b}} \text{ and } Q = rG$$
in the BKFR-Algorithm. In this case, one can easily check that $\gamma = |U_b - \frac{1}{U_b}| / \left(U_b + \frac{1}{U_b}\right)$. A specific algorithm for computing the best Gabor upper bound is proposed in [9].
- If the Gabor atom g is well concentrated, the associated Gabor matrix S is usually diagonal dominant. In this case, we perform the algorithm 1 with $S_d = D^{-1} * S$ instead of S, where D is a diagonal matrix with the same diagonal elements as S. We call this special algorithm **modified BKFR-method.**
- Algorithm 1 also gives a new way of computing the inverse of the Gabor frame operator.

■

3. NUMERICAL RESULTS

In this section we present some of our numerical results. The experiments shows that the Algorithm presented here has the big advantage of dealing with arbitrary Gabor signals even with huge signal length efficiently. All the numerical experiments were carried out using MATLAB 4.0 on a SUN-Station (COMPstation$^{\text{TM}}$40).

Figure 1 shows the comparison of convergence rates to compute the corresponding dual Gabor to the illustrated Gabor atom between the BKFR-method and CG-method presented in [7, 8].

Figure 2 shows some Gabor atom and the associated dual Gabor atom; the original Chirp-signal and the reconstructed one. The signal length is $N = 5120$ and the lattice constants are $(a, b) = (32, 10)$. The reconstruction relative error in this case is in order of 10^{-13}, which can be considered as an error-free reconstruction for all practical purposes.

Figure 3 shows the comparison of convergence rates of the modified BKFR-method, the BKFR-method and the CG-method. The signal length N is 2400 and the lattice constants (a, b) are $(40, 25)$. The reconstruction relative error is about 10^{-15}.

4. CONCLUSION

We have introduced a class of Gabor-type matrices and have shown that it is closed under the usual matrix multiplication. Based on this fact we have proposed an algorithm to perform the Gabor-type matrix multiplication through a special form of block-multiplication. As an application to discrete Gabor analysis we present the $BKFR$-algorithm to compute the (dual) Gabor analysis window iteratively, usually requiring only a few iterations.

5. ACKNOWLEDGEMENTS

The first author would like to thank the Department of Mathematics of the University of Vienna/AUSTRIA for its hospitality. The main part of the paper was prepared while he was in Vienna, with partial support from project PHY-08784 of the Austrian Science foundation FWF.

6. REFERENCES

[1] E. Bozzo, *Matrix Algebra and Discrete Transform*, Ph.D, Thesis: TD-1/94, Universita di Pisa.

[2] A. Bojańczyk, *Complexity of Solving Linear System in Different Models of Computation*, SIAM J. Numer. Anal. 21(3), 591–603, 1984.

[3] H. G. Feichtinger, C. Cenker and H. Steier, *Fast Iterative and Non-iterative Reconstruction Methods in Irregular Sampling*, Proc. ICASSP'91, 1773–1776, 1991.

[4] S. Li and D. M. Healy Jr, *On Pseudo Frame Decompositions and A Class of Discrete Gabor Expansions*, submitted to IEEE Trans. on Signal Processing.

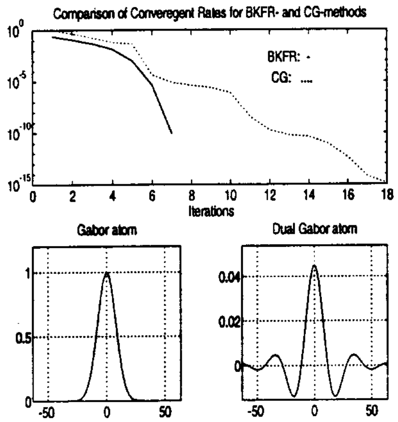

Figure 1: Convergent rates Comparison between CG-method and BKFR-method.

[5] R. S. Orr, *The Order of Computation for Finite Discrete Gabor Transform*, IEEE Trans. on Signal Processing, 41(2), 122–130, 1993.

[6] S. Qian and D. Chen, *Discrete Gabor Transform*, IEEE Trans. on Signal Processing, 41(7), 2429–2438, 1993.

[7] S. Qiu and H. G. Feichtinger, *The Structure of the Gabor Matrix and Efficient Numerical Algorithms for Discrete Gabor Expansions*, SPIE-Proceeding: VCIP'94 in Chicago, 1146–1157.

[8] S. Qiu, *Structural Properties of Gabor Transforms and Numerical Algorithms*, to be submitted.

[9] S. Qiu, *Attractive Discrete Huge Gabor Representation and Gabor-type Matrix Algebra*, to be submitted.

[10] R. Tolimieri and R. S. Orr, *Poisson Summation, the Ambiguity Function and the Theory of Weyl-Heisenberg Frames*, Preprint.

[11] J. Wexler and S. Raz, *Discrete Gabor Expansion*, Signal Processing, 21(1990), 207–220.

[12] M. Zibulski and Y. Y. Zeevi, *Oversampling in the Gabor Scheme*, IEEE Trans. on Signal Processing, 41(8), 2670–2687, August, 1993.

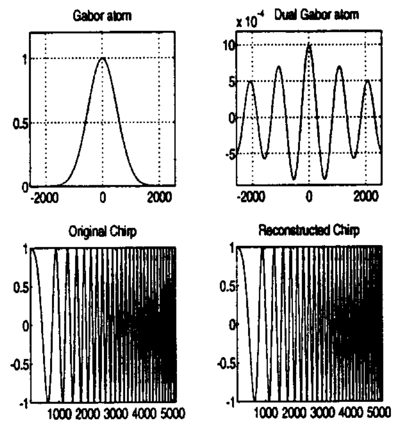

Figure 2: Gabor atom, the associated dual Gabor atom; the original Chirp-signal and the reconstructed Chirp. The signal length $N = 5120$ and the lattice constants $(a, b) = (32, 10)$.

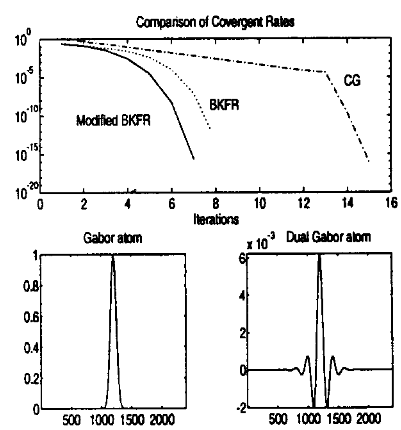

Figure 3: Convergent rates Comparison between CG-method, BKFR-method and modified BKFR-method.