NEW SAMPLING THEOREMS FOR MRA SUBSPACES

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ABSTRACT

In this paper, the existing sampling theory for MRA subspaces is generalized to several more cases. We consider derivative sampling, multiband sampling and sampling of wide sense stationary (WSS) random processes. We also show that the synthesizing functions form a Riesz basis for the corresponding MRA subspace.

1. INTRODUCTION

It is well-known that spaces of bandlimited functions are MRA subspaces. Walter [1] showed that other MRA subspaces share some properties of spaces of bandlimited functions. Namely, under very mild conditions on the scaling function $\phi(t)$, any $f(t) \in V_0 = \overline{\text{span}\ \{\phi(t-n)\}}$ can be recovered in a stable way from its integer samples $\{f(n)\}$. Janssen [2] extended Walter's results for uniform, but not necessarily integer sampling. He showed how any $f(t) \in V_0$ can be recovered from $\{f(n+a)\}$. In both of the above cases, there exists a function $S(t)$ such that $f(t) \in V_0$ can be recovered from its samples as follows

$$f(t) = \sum_n f(t_n) S(t-n) \tag{1.1}$$

The problem with these schemes is that $S(t)$ is guaranteed to be infinitely supported if the scaling function $\phi(t)$ is compactly supported (except for a few trivial cases). In [3] authors showed that if restricted to orthonormal MRA, then the Haar MRA is the only case where we can have both $\phi(t)$ and $S(t)$ compactly supported. In (1.1) all synthesis functions are shifted versions of a single function $S(t)$. If we allow more general synthesis functions, it can be shown that compact support is attainable. We will see an example of this in Sec. 2. In [4], compactly supported scaling and synthesizing functions were achieved by periodically nonuniform sampling. It was also shown that reconstruction from local averages and oversampling offer some additional nice features.

In this paper, we develop some further extensions. Namely, we consider:

1. Derivative sampling (section 2.2);
2. Multiband sampling (section 2.3);
3. Sampling of WSS random processes (section 2.4).

It can be shown that the synthesizing functions form a Riesz basis.

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2. FURTHER EXTENSIONS

In this section, we continue work done in [4]. Before going into derivations, we will introduce the notation and make clear our assumptions.

2.1. Assumptions and notation

We assume that $\{\phi(t-n)\}$ forms a Riesz basis for $V_0 \subset L^2(\mathcal{R})$. We use Janssen's assumption [2], namely, that $\phi(t)$ is bounded and that

$$\sum_n |\phi(t-n)| < C_\phi \tag{2.1}$$

converges uniformly on $[0,1]$. Since $\{\phi(t-n)\}$ is a Riesz basis for V_0, then for any $f(t) \in V_0$, there exists a unique sequence $\{c_n\} \in l^2$ such that [uniform convergence is insured by (2.1)]

$$f(t) = \sum_k c_k \phi(t-k). \tag{2.2}$$

If the sampling times are $t_n = n + u_n$, $0 \le u_n < 1$, then

$$f(t_n) = \sum_k c_k \phi(t_n - k) = \sum_k c_k \phi(u_n + n - k)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} C(e^{j\omega}) \Phi_{u_n}(e^{j\omega}) e^{jn\omega} d\omega, \tag{2.3}$$

where $\Phi_{u_n}(e^{j\omega}) = \sum_k \phi(u_n + k) e^{-jk\omega}$. The last expression will be used in all the derivations.

2.2. Derivative sampling

It is well known that a bandlimited function $f(t)$ can be recovered from samples of $f(t)$ and its derivative at *half* the Nyquist rate (see [5] for further references). In this subsection, we want to show how this can be extended for the case of wavelet subspaces. Let us demonstrate the idea on the example of reconstruction of $f(t) \in V_0$ from the samples of $f(t) = f_0(t)$ and its derivative $f'(t) = f_1(t)$ at rate 1/2.

We assume that the scaling function $\phi(t) = \phi_0(t)$ is compactly supported, and that it has a derivative $\phi'(t) = \phi_1(t)$, which satisfies Janssen's conditions stated in subsection 2.1. Consider uniform sampling, i.e., $t_n = n + u$. The above assumptions enable us to differentiate (2.2) term by

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term and get

$$f_1(t_n) = \sum_k c_k \phi_1(t_n - k) \tag{2.4}$$

Using (2.3), we have

$$f_i(t_n) = \int_{-\pi}^{\pi} e^{jn\omega} C(e^{j\omega}) \Phi_i(e^{j\omega}) \frac{d\omega}{2\pi}, \tag{2.5}$$

for $i = 0, 1$, where

$$\Phi_i(e^{j\omega}) = \sum_n \phi_i(u+n) e^{-jn\omega}. \tag{2.6}$$

From [2], we know that $f(t)$ can be reconstructed from $\{f_0(t_n)\}$. This means that the sequence of derivatives $\{f_1(t_n)\}$ is redundant. The idea is to use this redundancy to reconstruct $f(t)$ from subsampled sequences $\{f_0(t_{2n})\}$ and $\{f_1(t_{2n})\}$. Notice that this scheme corresponds to a two-channel maximally decimated filter bank with $\Phi_0(e^{j\omega})$ and $\Phi_1(e^{j\omega})$ as analysis filters. The situation is shown in Fig. 2.1.

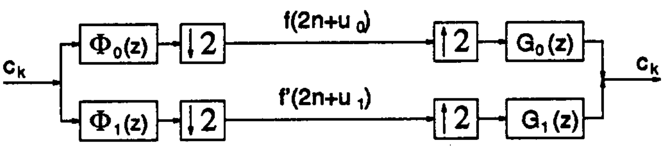

Fig. 2.1. An interpretation of derivative sampling.

The following theorem provides sufficient conditions for a stable reconstruction.

Theorem 1. Let $\mathbf{E}(z)$ be the polyphase matrix of analysis filters in Fig. 2.1. Any $f(t) \in V_0$ can be recovered from samples $\{f(t_{2n})\}$ and $\{f'(t_{2n})\}$ in a stable way if det $\mathbf{E}(e^{j\omega}) \neq 0$ for all $\omega \in [-\pi, \pi]$.

If the theorem is satisfied, $f(t)$ can be reconstructed by

$$f(t) = \sum_n f(t_{2n}) S_0(t-2n) + \sum_n f'(t_{2n}) S_1(t-2n),$$

where $S_0(t)$ and $S_1(t)$ have Fourier transforms given by $G_0(e^{j\omega})\Phi(\omega)$ and $G_1(e^{j\omega})\Phi(\omega)$ respectively. These synthesizing functions form a Riesz basis for the space they span (V_0 in this case). This can be concluded from the properties of the synthesis polyphase matrix.

The above example can be easily generalized to the case of higher derivatives. Assume that the scaling function $\phi(t)$ and its $M-1$ derivatives satisfy Janssen's conditions from subsection 2.1. Then $f(t)$ can be reconstructed from the samples of $f(t)$ and its $M-1$ derivatives at $1/M^{th}$ Nyquist rate, provided conditions of Theorem 1 are satisfied. Synthesizing functions can be constructed as in [6]. Let us illustrate the above derivations on the case of quadratic splines.

Example 2.1. Consider the MRA generated by the quadratic splines. The scaling function is

$$\phi(t) = \begin{cases} t^2/2, & \text{for } 0 \le t < 1, \\ -(t-3/2)^2 + 3/4, & \text{for } 1 \le t < 2, \\ \frac{1}{2}(t-3)^2, & \text{for } 2 \le t < 3, \\ 0, & \text{otherwise.} \end{cases}$$

$\phi(t)$, $\phi'(t)$ and integer samples are shown in Fig. 2.2.

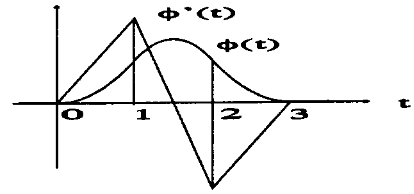

Fig. 2.2. $\phi(t)$, $\phi'(t)$ and its samples at integers.

From the figure it is easy to see that $\Phi_0(z) = \sum_n \phi(n) z^{-n} = 1/2(z^{-1} + z^{-2})$ and $\Phi_1(z) = \sum_n \phi'(n) z^{-n} = z^{-1} - z^{-2}$. $\mathbf{E}(z) = \begin{pmatrix} z^{-1}/2 & 1/2 \\ -z^{-1} & 1 \end{pmatrix}$ and $\mathbf{R}(z) = \begin{pmatrix} z & -z/2 \\ 1 & 1/2 \end{pmatrix}$, so that the FIR synthesis filters are $G_0(z) = z + z^2$ and $G_1(z) = z/2 - z^2/2$. Finally, synthesis functions are $S_0(t-2n)$ and $S_1(t-2n)$, where $S_0(t) = \phi(t+1) + \phi(t+2)$ and $S_1(t) = \frac{1}{2}\phi(t+1) - \frac{1}{2}\phi(t+2)$.

2.3. Multiband sampling

Consider a signal $F(\omega)$ as in Fig. 2.3. If $F(\omega)$ is regarded as a lowpass signal, minimum necessary sampling rate is $2\omega_s$. If it is regarded as a bandpass signal, it can be verified that aliasing copies $F(\omega + k\omega_s)$ caused by sampling at the rate ω_s do not overlap. Therefore, minimum sampling rate in this case is ω_s.

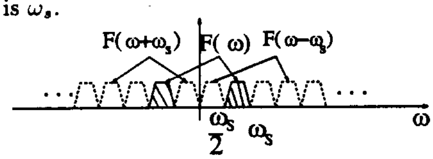

Fig. 2.3. An ideal bandpass signal and its aliasing copies.

The aim of this subsection is to find what the equivalent of this situation in wavelet subspaces is.

Spaces V_k, roughly speaking, contain lowpass signals, whereas W_k spaces contain bandpass signals. So far, we have examined lowpass signals only (ones that belong to V_0). We will show that if a signal does not occupy the whole frequency range that V_0 covers, it can be sampled at a lower rate. Assume that $f(t) \in W_{-1} \dot{+} W_{-2} \dot{+} \cdots \dot{+} W_{-J} \subset V_0$. This means that there are sequences $\{c_{-1,n}\}, \{c_{-2,n}\}, \ldots, \{c_{-J,n}\} \in l^2$ such that

$$f(t) = \sum_{k=-J}^{-1} \sum_n c_{k,n} 2^{k/2} \psi(2^k t - n). \tag{2.7}$$

From Walter's work, we know that since $f(t) \in V_0$, it can be recovered from its integer samples. Here, the aim is to exploit the fact that $f(t)$ belongs to a subspace of V_0 and sample it at a lower rate. The idea is to find an invertible map from the sequences $\{c_{k,n}\}$ to a sequence of samples of $f(t)$. For this, let us sample $f(t)$ at $t_{k,n} = n2^{-k} + u_k$, $k = -J, -J+1 \ldots, -1$, where $u_k \in [0, 2^{-k})$. Intuitively, this rate should be enough, since we can project $f(t)$ onto each of W_k's and then sample those projections at rates 2^k. In order to simplify the analysis that follows, let us find some equivalent system where all the inputs/outputs operate at the same rate. For this, let

$$C_k(e^{j\omega}) = \sum_n c_{k,n} e^{-j\omega n} \quad \text{and} \quad F_m(e^{j\omega}) = \sum_n f(t_{m,n}) e^{-j\omega n} \tag{2.8}$$

be the Fourier transforms of $\{c_{k,n}\}$ and $\{f(t_{m,n})\}$. In order to bring all these signals to the same rate, we expand $C_k(e^{j\omega})$'s and $F_k(e^{j\omega})$'s into their 2^{J+k}-fold polyphase components

$$C_k(e^{j\omega}) = \sum_{l=0}^{2^{J+k}-1} e^{-j\omega l} C_{k,l}(e^{j\omega 2^{J+k}})$$

and

$$F_k(e^{j\omega}) = \sum_{l=0}^{2^{J+k}-1} e^{-j\omega l} F_{k,l}(e^{j\omega 2^{J+k}}), \qquad -J \le k \le -1. \tag{2.9}$$

Now that all the inputs and outputs are brought to the same rate, our system can be represented as in Fig. 2.4.

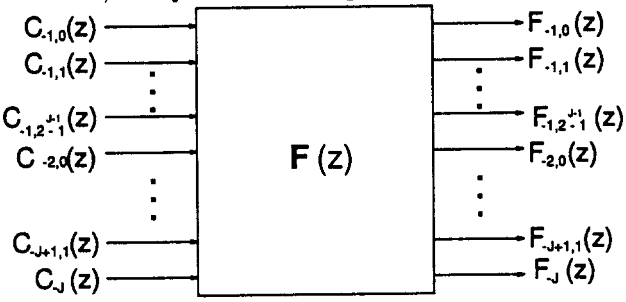

Fig. 2.4. An interpretation of multiband sampling.

The entries of $\mathbf{F}(z)$ in Fig. 2.4. are functions of the chosen MRA and sampling points. Detailed derivations can be found in [6]. An MIMO version of the Wiener's theorem [7] gives us sufficient conditions for the existence of a stable inversion scheme.

Theorem 2. If a function $f(t) \in W_{-i_1} \dot{+} W_{-i_2} \dot{+} \cdots \dot{+} W_{-i_J} \subset 0$ $(1 \le i_1 < i_2 < \ldots < i_J)$ is sampled at the rate $2^{-i_1}+2^{-i_2}+\cdots+2^{-i_J} < 1$, then there exists a stable reconstruction scheme if $\mathbf{F}(e^{j\omega})$, as defined above, is nonsingular for all $\omega \in [-\pi, \pi]$.

Remark. Notice that if the projection of $f(t)$ onto some of W_k's is zero, we can drop the corresponding term $C_k(e^{j\omega})$ and sample at an even lower rate.

In this case, there will be $2^J - 1$ functions $S_k(t)$ obtained from $\mathbf{F}^{-1}(z)$ and $\Phi(\omega)$, and the synthesis functions are $S_k(t - 2^J n)$, for $n \in \mathcal{Z}$ (see [6] for details). Those synthesis functions form a Riesz basis as well.

2.4. Sampling of WSS random processes

The problem of sampling of random processes was thoroughly investigated by the end of 1960's. Uniform and nonuniform sampling of WSS and nonstationary bandlimited random processes was considered (for an overview, see [5]). In this section we want to look at the problem of uniform and nonuniform sampling of WSS random processes related to wavelet subspaces.

Let $\phi_a(t)$ be the deterministic autocorrelation function of $\phi(t)$, i.e., $\phi_a(\tau) = \int \phi(t+\tau)\phi^*(t)dt$. We keep assumptions from Sec. 2, namely that $\{\phi(t-n)\}$ is a Riesz bases for $V_0 = \overline{\text{span}\{\phi(t-n)\}}$ and that $\phi(t)$ satisfies Janssen's conditions from subsection 2.1. We will consider random processes whose autocorrelation functions have the following form

$$R_{ff}(t) = \sum_n c_n \phi_a(t-n), \tag{2.10}$$

where $\{c_n\} \in l^1$. The PSD function has the following form

$$S_{ff}(\omega) = C(e^{j\omega})\Phi_a(\omega), \tag{2.11}$$

where $C(e^{j\omega}) \ge 0$ and $\Phi_a(\omega) = |\Phi(\omega)|^2$.
We will consider both uniform and nonuniform sampling.

Uniform sampling

Consider a random processes $\{f(t), -\infty < t < \infty\}$ with autocorrelation functions $R_{ff}(\tau)$ of the form (2.10). The discrete parameter autocorrelation function is

$$r_{ff}(m,n) = E[f(n+m)f^*(m)] = R_{ff}(n), \tag{2.12}$$

and therefore, $\{f(n)\}$ is a discrete parameter WSS random process.

Let $s_{ff}(e^{j\omega}) = \sum_n S_{ff}(\omega + 2\pi n)$. Then the Fourier coefficients of $s_{ff}(e^{j\omega})$ are integer samples of the autocorrelation function $r_{ff}(n) = R_{ff}(n)$.

First, we show that the random process $\{f(t)\}$ cannot be reconstructed from the samples $\{f(n)\}$, if the synthesizing functions are restricted to be shifts of one function (unless, of course, the random process is bandlimited). In order to show this, assume the contrary. Let there be a function $g(t) \in L^2(\mathcal{R})$ such that $\{f(t)\}$ is equal to $\sum_n f(n)g(t-n)$ in MS sense. The error random process

$$e(t) = f(t) - \sum_n f(n)g(t-n) \tag{2.13}$$

has autocorrelation function

$$R_{ee}(t,\tau) = E\left[e(t)e^*(t-\tau)\right]. \tag{2.14}$$

It is easy to check that $R_{ee}(t+1,\tau) = R_{ee}(t,\tau)$, so the error is a cyclo-WSS random process with period $T = 1$. We can average $R_{ee}(t,\tau)$ over T, to get the autocorrelation function $R_{ee}(\tau)$. Its variance $\sigma^2 = R_{ee}(0)$ is

$$R_{ff}(0) - \int R_{ff}(t-n)g^*(t-n)dt - \int R_{ff}(n-t)g(t-n)dt + $$
$$+ \sum_l R_{ff}(l) \int g(t-l)g^*(t)dt. \tag{2.15}$$

Using Parseval's identity, the above expressions can be simplified to

$$\frac{1}{2\pi}\int S_{ff}(\omega)\left(1 - G^*(\omega) - G(\omega) + \sum_k |G(\omega+2\pi k)|^2\right)d\omega \tag{2.16}$$

This cannot be zeroed for any choice of $G(\omega)$ unless $\phi(t)$ is bandlimited.

Even though we cannot recover the random process in MS sense, we can still recover the PSD function $S_{ff}(\omega)$. We know that $r_{ff}(n) = R_{ff}(n)$ and that $s_{ff}(e^{j\omega}) = \sum_k S_{ff}(\omega + 2\pi k)$. Substituting the special form of $S_{ff}(\omega)$ into the last formula, we get

$$s_{ff}(e^{j\omega}) = C(e^{j\omega}) \sum_k |\Phi(\omega + 2\pi k)|^2. \tag{2.17}$$

Since $s_{ff}(e^{j\omega}) = \sum_n r_{ff}(n)e^{-jn\omega}$, we can recover $C(e^{j\omega})$ as follows

$$C(e^{j\omega}) = \frac{\sum_n r_{ff}(n)e^{-jn\omega}}{\sum_k |\Phi(\omega + 2\pi k)|^2}. \tag{2.18}$$

Notice that the division is legal, because of the following reason: $\{\phi(t-n)\}$ is assumed to form a Riesz basis for its span. Therefore, there are constants $0 < A \leq B < \infty$ such that $A \leq \sum_k |\Phi(\omega + 2\pi k)|^2 \leq B$ a.e., and the result of the division is in $L^2[-\pi, \pi]$. Finally, the reconstructed spectrum is

$$S_{ff}(\omega) = \frac{\sum_n r_{ff}(n)e^{-jn\omega}}{\sum_k |\Phi(\omega + 2\pi k)|^2} |\Phi(\omega)|^2. \tag{2.19}$$

Remark. If $\{\phi(t-n)\}$ forms an orthonormal basis, then $\sum_k |\Phi(\omega + 2\pi k)|^2 = 1$ a.e., and the above equation simplifies to

$$S_{ff}(\omega) = |\Phi(\omega)|^2 \sum_n r_{ff}(n)e^{-jn\omega}. \tag{2.20}$$

Nonuniform sampling

It can be easily seen that a deterministic nonuniform sampling of a random process produces a nonstationary discrete parameter random process. In order to preserve stationarity, we introduce randomness into the sampling times (i.e., jitter). These, so called stationary point random processes, were investigated in [8]. One special case is when the sampling times are $t_n = n + u_n$, where u_n are independent random variables with some distribution function $p(u)$. Let $\gamma(\omega) = E_u[e^{-ju\omega}]$ be its characteristic function. The autocorrelation sequence of the discrete parameter random process $\{f(t_n)\}$ is

$$r_{ff}(m,n) = E_u\left[E[f(t_{n+m})f^*(t_m)]\right] = E_u[R_{ff}(t_{m+n} - t_m)] =$$

$$= \frac{1}{2\pi} \int S_{ff}(\omega) E_u[e^{j(u_{n+m} - u_m)\omega}] e^{jn\omega} d\omega. \tag{2.21}$$

Since u_n's are mutually independent, we have

$$E_u[e^{ju_{n+m}\omega} e^{-ju_m\omega}] = \begin{cases} |\gamma(\omega)|^2, & \text{if } n \neq 0 \\ 1, & \text{if } n = 0 \end{cases},$$

so that we finally get

$$r_{ff}(m,n) = \begin{cases} \frac{1}{2\pi} \int e^{jn\omega} |\gamma(\omega)|^2 S_{ff}(\omega) d\omega, & \text{if } n \neq 0; \\ \frac{1}{2\pi} \int S_{ff}(\omega) d\omega & \text{if } n = 0. \end{cases} \tag{2.22}$$

Since $r_{ff}(m,n)$ is independent of m, $\{f(t_n)\}$ is a WSS random process and we will just leave out index m in (2.22). Our PSD function has a special form $S_{ff}(\omega) = C(e^{j\omega})|\Phi(\omega)|^2$, and putting this in (2.22), we get

$$r_{ff}(n) = \frac{1}{2\pi} \int e^{jn\omega} |\gamma(\omega)|^2 C(e^{j\omega}) |\Phi(\omega)|^2 d\omega =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{jn\omega} C(e^{j\omega}) \sum_k |\gamma(\omega + 2\pi k)|^2 |\Phi(\omega + 2\pi k)|^2 d\omega, \tag{2.23}$$

for $n \neq 0$. First, we recover $C(e^{j\omega})$ as in the case of uniform sampling. Then the original PSD function is

$$S_{ff}(\omega) = \frac{r(0) + \sum_{n \neq 0} r_{ff}(n)e^{-jn\omega}}{\sum_k |\gamma(\omega + 2\pi k)|^2 |\Phi(\omega + 2\pi k)|^2} |\Phi(\omega)|^2, \tag{2.24}$$

where $r(0)$ can be found from (2.22) and (2.23).

In summary, given a WSS random process, its autocorrelation function can be estimated using formulas (2.20) and (2.24). If the autocorrelation function does not satisfy condition (2.10), there will be some aliasing error.

4. CONCLUSION

We extended the sampling theory for MRA subspaces to derivative and multiband sampling and sampling of WSS random processes. Riesz basis property of the synthesizing functions was established.

REFERENCES

[1] Walter, G. G. "A sampling theorem for wavelet subspaces", *IEEE Tr. IT*, vol. 38, pp. 881-884, March 1992.

[2] Janssen, A. J. E. M. "The Zak transform and sampling theorems for wavelet subspaces", *IEEE Tr. SP*, vol. 41, pp. 3360-3365, December 1993.

[3] X. Xia, and Z. Zhang "On sampling theorem, wavelets, and wavelet transform", *IEEE Tr. SP*, vol. 41, pp. 3524-3536, December 1993.

[4] Djokovic, I. and P. P. Vaidyanathan "Nonuniform sampling/ oversampling and extensions", IEEE-SP Int. Symp. TF/TS Analysis, Philadelphia, Oct. 1994.

[5] Jerri, A. J. "The Shannon sampling theorem-Its various extensions and applications: a tutorial review", *Proc IEEE* vol. 65, pp. 1565-1596, 1977.

[6] Djokovic, I., and P. P. Vaidyanathan "Generalized sampling theorems in multiresolution subspaces", Caltech rep. Aug. 1994.

[7] Vidyasagar, M. *Control system synthesis – a factorization approach*, MIT press, Cambridge, 1985.

[8] Beutler, F. J., and Leneman, D. A. "The theory of stationary point processes", *Acta Math*, vol. 116, pp. 159-197, September 1966.