SHIFT INVARIANT WAVELET PACKET BASES

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ABSTRACT

In this work, a shifted wavelet packet (SWP) library, containing all the time shifted wavelet packet bases, is defined. A corresponding shift-invariant wavelet packet decomposition (SIWPD) search algorithm for a "best basis" is introduced. The search algorithm is representable by a binary tree, in which a node symbolizes an appropriate subspace of the original signal. We prove that the resultant "best basis" is orthonormal and the associated expansion, characterized by the lowest "information cost", is shift-invariant. The shift-invariace stems from an additional degree of freedom, generated at the decomposition stage, and incorporated into the search algorithm. We prove that for any subspace it suffices to consider one of two alternative decompositions, made feasible by the SWP library. The computational complexity of SIWPD may be controlled at the expense of the attained information cost, to an extent of $O(2Nlog_2N)$.

1. INTRODUCTION

Wavelet packets (WP) were first introduced by Coifman and Meyer [1] as a library of orthogonal bases for $L^2(R)$. Implementation of a "best-basis" selection procedure for a prescribed signal (or a family of signals) requires the introduction of an acceptable "cost function" which translates "best" into a minimization process [2]. The cost function selection is intimately related to the specific nature of the application at hand. Entropy, for example, may constitute a reasonable choice if signal compression, identification or classification are the applications of interest. Statistical analysis of the "best basis" coefficients may be used as a signature, representing the original signal. A major deficiency of such an approach has to do with the badly lacking property of shift-invariance. Both, the wavelet packet decomposition (WPD) of Coifman and Wickerhauser as well as the extended algorithm proposed by Herley et al. [3], are sensitive to the signal location with respect to the chosen time origin.

Shift-invariant multiresolution representations exist. However, these methods either entail high oversampling rates [4] or alternatively, the resulting representations are non-unique [5]. Mallat and Zhang [6] have suggested an adaptive "matching pursuit" algorithm. Under this approach the retainment of shift-invariance, necessitates an oversized library containing the basis functions and all their shifted versions. The obvious drawbacks of "matching pursuit" are the rather high complexity level as well as the non-orthogonality of the expansion. In another approach [7], shift-invariance is achieved by an adaptive translation of subbands. This translation, merely determined by one level decomposition, leads to a sub-optimal representation, which is a special case in the context of the present work.

In this work we generate a shifted wavelet packet library and introduce a shift invariant wavelet packet decomposition (SIWPD) algorithm for a "best basis" selection with respect to an additive cost function (e.g., entropy). We prove that the proposed algorithm leads to a "best basis" representation that is both shift-invariant and orthogonal. To demonstrate the shift invariant properties of SIWPD, compared to WPD which lacks this feature, we refer to the expansions of the signals depicted in Figs. 1a, 2a (the signals at hand are identical to within a time-shift). For definiteness, we choose D_2 [8] to serve as the "scaling function" and entropy as the cost function. Figs. 1b, 2b and Figs. 1c, 2c depict the "best basis" expansion of the respective signals under the WPD and the SIWPD algorithms. A comparison of Fig. 1 and Fig. 2 readily reveals the sensitivity of WPD to temporal shifts while the "best basis" SIWPD representation is indeed shift-invariant.

It should be mentioned that under SIWPD and in contrast to WPD, the "best basis" expansion is also characterized by the invariance of the "information cost". This feature is significant as it facilitates a meaningful quantitative comparison between alternative shifted WP libraries. Usually such a comparison between alternative libraries lacks meaning for WP as demonstrated by the example, summarized in Table 1. Here, the entropies of the signals g_1 and g_2 (Figs. 1a and 2a, respectively) are compared. The expansions are on the "best bases" stemming from both the WPD and SIWPD algorithms and for D_2 and C_1 (Coiflet with two vanishing moments [9]) scaling functions. We can readily observe the shift-invariance under SIWPD and the fact that the selection of C_1 is consistently advantageous over D_2. Just as obvious, is the futility of attempting a comparison between the C_1 and D_2 based libraries under WPD. D_2 is better for g_1 while C_1 is advantageous in representing g_2 (which, as we recall, is just a shifted version of g_1).

	WPD		SIWPD	
	g_1	g_2	g_1	g_2
D_2	1.77	2.15	1.67	1.67
C_1	2.30	1.80	1.55	1.55

Table 1: Entropies of g_1 and g_2 represented on "best bases" obtained via WPD and SIWPD using libraries derived from C_1 and D_2 scaling functions.

2. THE SHIFTED WAVELET PACKET LIBRARY

Let $\{\psi_n(x)\}$ be a wavelet packet family [1], and introduce the notation

$$B^j_{\ell,n,m} \triangleq \left\{2^{(\ell+j)/2}\psi_n\left[2^\ell(2^j x - m) - k\right] \,:\, k \in \mathbb{Z}\right\} \quad (1)$$

$$U^j_{\ell,n,m} = \text{clos}_{L^2(\mathbb{R})}(B^j_{\ell,n,m}) \quad (2)$$

The energy density of $\psi_n[2^\ell(2^j x - m) - k]$ (when a proper "scaling function" is selected) is concentrated about the nominal point $2^{-j}(2^{-\ell}k+m)$, has an effective support range $\approx 2^{-\ell-j}$ about this point and is roughly characterized by n oscillations.

Definition 1 *A "shifted-wavelet-packet" (SWP) library is the collection of all the orthonormal subsets of*

$$\left\{B^j_{\ell,n,m} \,:\, \ell \in \mathbb{Z}_-,\, n \in \mathbb{Z}_+,\, 0 \le m < 2^{-\ell}\right\}.$$

Proposition 1 *Let $E = \{(\ell, n, m)\} \subset \mathbb{Z}_- \times \mathbb{Z}_+ \times \mathbb{Z}_+$, $0 \le m < 2^{-\ell}$, denote a collection of indexes satisfying*

(i) *The segments $[2^\ell n,\, 2^\ell(n+1))$ are a disjoint cover of $[0, 1)$.*

(ii) *For all $(\ell_1, n_1, m_1), (\ell_2, n_2, m_2) \in E$, the relation*

$$[2^{\ell_1}n_1, 2^{\ell_1}(n_1+1)) \subset [2^{\ell_{2p}}n_{2p}, 2^{\ell_{2p}}(n_{2p}+1)) \quad (3)$$

where $\ell_{2p} = \ell_2 + 1$ and $n_{2p} = n_2\, div\ 2$, implies [1]

$$m_1 \mod 2^{-\ell_2} = m_2 . \quad (4)$$

Then E generates an orthonormal (ON) basis for $V_j \equiv U^j_{(0,0,0)}$, i.e. $\bigcup_{(\ell,n,m)\in E} B^j_{\ell,n,m}$ is an ON basis, and the set of all E as specified above generates a SWP library.

The proof is detailed in [10]. The SWP library thus created is larger than the WP library proposed in [1] by a factor $2N/\log_2 N$. The SWP library can still be cast into a tree configuration, where each node is indexed by (ℓ, n, m) and represents the subspace $U^j_{(\ell,n,m)}$. Condition (ii) above is equivalent to demanding that the relative shift between a prescribed "parent" node and all its "children" nodes is necessarily a constant whose value is restricted to either zero or to $2^{-\ell_p}$ (ℓ_p is the parent's level). The tree configuration facilitates an efficient "best basis" selection process. However, in contrast to [2] the "best basis" is now shift-invariant.

3. THE BEST BASIS SELECTION

Let $f \in V_j$, let $\mathcal{M}$ denote an additive cost function and let $\mathcal{B}$ represent a SWP library.

Definition 2 *The best basis for f in $\mathcal{B}$ with respect to $\mathcal{M}$ is $B \in \mathcal{B}$ for which $\mathcal{M}(Bf)$ is minimal. Here, $\mathcal{M}(Bf)$ is the information cost of representing f on the basis $B \in \mathcal{B}$.*

[1] $x\ div\ y$ denotes the integer part of the ratio x/y, and $x \bmod y$ represents its remainder.

Let $A^j_{\ell,n,m}$ denote the "best basis" for the subspace $U^j_{\ell,n,m}$. Accordingly, $A^j_{0,0,0}$ constitutes the "best basis" for f with respect to $\mathcal{M}$. Henceforth, for notational simplicity, we omit the fixed index j. The desired "best basis" can be determined recursively by setting

$$A_{\ell,n,m} = \begin{cases} B_{\ell,n,m} \text{ if } & \mathcal{M}(B_{\ell,n,m}f) \\ & \le \sum_{i=0}^{1}\mathcal{M}(A_{\ell-1,2n+i,m_c}f) \\ A_{\ell-1,2n,m_c} \oplus A_{\ell-1,2n+1,m_c}, \text{ else} & \end{cases} \quad (5)$$

where

$$m_c = \begin{cases} m, \text{ if } & \sum_{i=0}^{1}\mathcal{M}(A_{\ell-1,2n+i,m}f) \\ & \le \sum_{i=0}^{1}\mathcal{M}(A_{\ell-1,2n+i,m+2^{-\ell}}f) \\ m + 2^{-\ell}, & \text{else} \end{cases} \quad (6)$$

The recursive sequence proceeds down to a specified level $\ell = -L$ ($L \le \log_2 N$), where

$$A_{-L,n,m} = B_{-L,n,m} . \quad (7)$$

The stated procedure resembles that proposed by Coifman and Wickerhauser [2] with an added degree of freedom facilitating a relative shift (i.e., $m_c \ne m$) between a "parent" node and its respective "children" nodes. It is re-emphasized that the recursion considered herein restricts the shift to one of two values ($m_c - m \in \{0, 2^{-\ell}\}$). Other values are unacceptable if the orthonormality of the "best basis" is to be preserved. As it turns out, the generated degree of freedom is crucial in establishing "time-invariance". The recursive sequence proposed in [2] may be viewed as a special case where $m_c - m$ is arbitrarily set to zero.

Proposition 2 *The best basis stemming from the previously described recursive algorithm is shift invariant.*

Proof: Let $f, g \in V_j$ where f and g are "identical to within a time-shift", i.e., $g(x) = f(x - q2^{-j})$. Let A_f and A_g denote the "best bases" for f and g, respectively. It can be shown [10] that

$$B_{\ell,n,m} \subset A_f \quad (8)$$

implies

$$B_{\ell,n,\tilde{m}} \subset A_g, \qquad \tilde{m} = (m+q) \bmod (2^{-\ell}) \quad (9)$$

for all $m, n \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}_-$. Hence, A_f and A_g are "identical to within a time-shift".

□.

An alternative view of SIWPD is facilitated via filter bank terminology [11]. Accordingly, each "parent" node is expanded by high-pass and low-pass filters, followed by a 2:1 down-sampling. In executing WPD, down-sampling is achieved by ignoring all even-indexed (or all odd-indexed) terms. In contrast, when pursuing SIWPD, the down sampling is carried out *adaptively* for the prescribed signal. That is, we retain either the odd or the even terms. The preferred choice is always the one that minimizes the cost function. The special case where, at any resolution level, only low frequency nodes are further expanded corresponds to a shift invariant wavelet transform (SIWT).

The SIWPD expansion generates an ordinary binary tree [2]. However, each generated branch is now designated by either fine or heavy lines (Fig. 3) depending on the adaptive selection of the odd or the even terms, respectively. it can be readily observed that, in contrast to WPD, SIWPD expansion leads to tree configurations that are independent of the time-origin. Fine and heavy lines may, however, exchange positions. (e.g., compare Fig. 1e and Fig. 2e).

4. THE INFORMATION-COST COMPLEXITY TRADEOFF

WPD lacks shift-invariance but is characterized by an attractive complexity level of $O(NL)$, where L denotes the lowest resolution level in the expansion tree. Comparatively, the complexity level, $O(N2^{L+1})$, associated with SIWPD is substantially higher. In return, one may achieve a potentially large reduction of the information cost, in addition to gaining the all important "shift-invariance". However, whenever the SIWPD complexity is viewed as intolerable, one may resort to a sub-optimal SIWPD procedure entailing a reduced complexity, and higher information cost while still retaining the desirable shift-invariance.

The "best basis" for $f \in V_j$ with respect to $\mathcal{M}$ is, once again, obtained recursively via (5), but contrary to the procedure of Section 3, now the selection of m_c does not necessitate parent-node tree expansion down to the lowest level. Let $C_{\ell,n,m,d}$ denote the best basis for $U_{\ell,n,m}$ subject to constraining the decomposition to a $(1+d_\ell)$-level tree, where

$$d_\ell = \begin{cases} d, & d-L \leq \ell \leq 0 \\ L+\ell, & \text{else} \end{cases} \tag{10}$$

and

$$1 \leq d \leq L.$$

m_c is then determined by

$$m_c = \begin{cases} m, & \text{if } \sum_{i=0}^{1} \mathcal{M}(C_{\ell-1,2n+i,m,d} f) \\ & \quad \leq \sum_{i=0}^{1} \mathcal{M}(C_{\ell-1,2n+i,m+2^{-\ell},d} f) \\ m+2^{-\ell}, & \text{else} \end{cases} \tag{11}$$

For $d=1$ or $\ell=-L$ we obtain $C_{\ell,n,m,d} = B_{\ell,n,m}$, while for $\ell > -L$ and $d > 1$ $C_{\ell,n,m,d}$ is obtained recursively according to

$$C_{\ell,n,m,d} = \begin{cases} B_{\ell,n,m} \\ C_{\ell-1,2n,m,d-1} \oplus C_{\ell-1,2n+1,m,d-1} \\ C_{\ell-1,2n,m+2^{-\ell},d-1} \oplus C_{\ell-1,2n+1,m+2^{-\ell},d-1} \end{cases} \tag{12}$$

where $C_{\ell,n,m,d}$ takes on that value which minimizes the cost function $\mathcal{M}$.

Since, at each level, the tree expansion is restricted to $d_\ell \leq d$ levels, the complexity is now $O[N2^d(L-d+2)]$. In the extreme case, $d=1$, the complexity, $O(2NL)$, is similar to that associated with WPD, and the representation merges with that proposed in [7]. Clearly, the larger d and L, the larger the complexity, however, the determined "best basis" is of a higher quality; namely, characterized by a lower "information cost".

5. CONCLUSION

A library of orthonormal Shifted Wavelet Packets is defined and a search algorithm leading to a Shift Invariant Wavelet Packet Decomposition (SIWPD) is introduced. When compared with the WPD algorithm proposed in [2], SIWPD is determined to be advantageous in three respects. First, it leads to a "best basis" that is shift-invariant. Second, the resulting decomposition is characterized by a lower information cost function. Third, the complexity is controlled at the expense of the information cost. These advantages may prove crucial to signal compression, identification or classification applications. Furthermore, the shift-invariant nature of the information cost, renders this quantity a characteristic of the signal for a prescribed wavelet packet library. It should be possible now to quantify the relative efficiency of various libraries (i.e., various "scaling function" selections) with respect to a given cost function.

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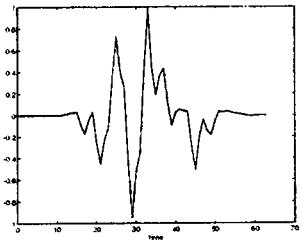

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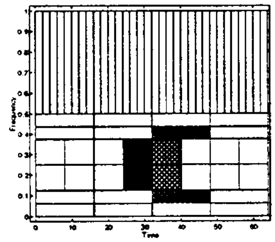

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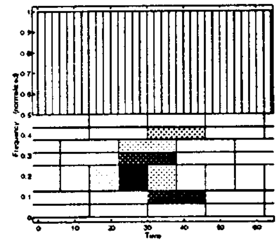

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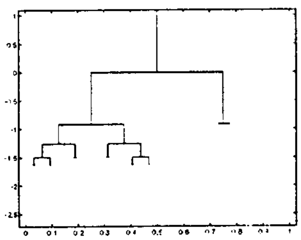

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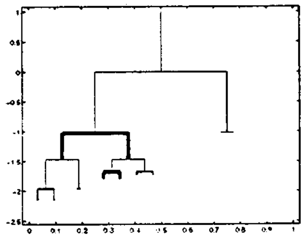

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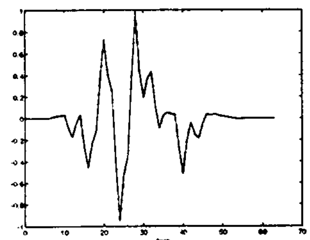

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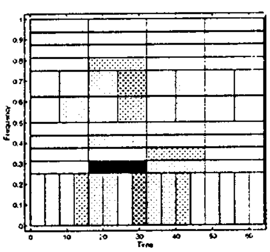

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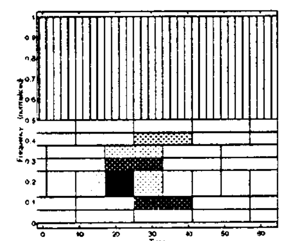

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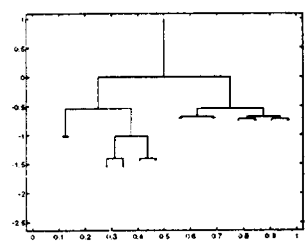

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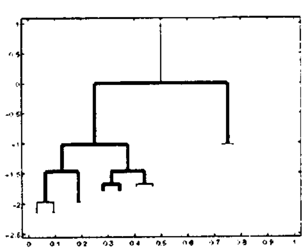

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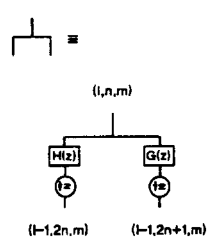

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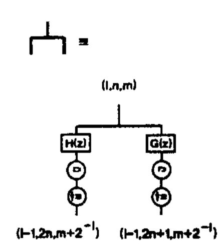

(3b)

Fig. 1: (a) $g_1(t)$. (b) A "best basis" WPD of g_1. (c) A "best basis" SIWPD of g_1. (d) WPD based expansion tree of g_1. (e) SIWPD based expansion tree of g_1.

Fig. 2: (a) $g_2(t)$. (b) A "best basis" WPD of g_2. (c) A "best basis" SIWPD of g_2. (d) WPD based expansion tree of g_2. (e) SIWPD based expansion tree of g_2.

Fig. 3: A "parent" node binary expansion according to SIWPD: (a) High and low-pass filtering followed by a 2:1 downsampling. (b) High and low-pass filtering followed by a time-shift (D) and subsequently by a 2:1 downsampling.