DISCRETE-TIME, DISCRETE-FREQUENCY TIME-FREQUENCY REPRESENTATIONS

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ABSTRACT

A discrete-time, discrete-frequency Wigner distribution is derived using a group-theoretic approach. It is based upon a study of the Heisenberg group generated by the integers mod N, which represents the group of discrete-time and discrete-frequency shifts. The resulting Wigner distribution satisfies several desired properties. An example demonstrates that it is a full-band time-frequency representation, and, as such, does not require special sampling techniques to suppress aliasing. It also exhibits some interesting and unexpected interference properties. The new distribution is compared with other discrete-time, discrete-frequency Wigner distributions proposed in the literature.

1. INTRODUCTION

The Wigner distribution is an important tool for analyzing signals. Its usefulness arises from the fact that it satisfies many desired mathematical properties. Such properties include the time and frequency marginals, Moyal's formula, the relationship with the ambiguity function, and the relationship with the spectrogram (and related smoothing). For practical applications involving discrete computations, though, the utility of the Wigner distribution depends greatly on the discretized form chosen for use.

There are several different approaches to extending the theory of continuous time-frequency distributions to the discrete-time case, [1, 2, 3, 4]. Unfortunately, a discrete-time, discrete-frequency Wigner distribution which satisfies all of the desired properties has not yet been determined. A common form used is a frequency-discretized version of the discrete-time Wigner distribution outlined by Claasen and Mecklenbräuker [5], which we call WCM. This time-frequency representation satisfies some of the desired properties: the auto-WCM Wigner distribution is real and the convolution of the WCM Wigner distribution of a signal with the WCM Wigner distribution of a window is the corresponding spectrogram, other properties, such as Moyal's formula and the marginals, are not satisfied. More significantly, this discretized version is a half-band representation, so that one must double the signal sampling rate to compute it properly. One would then like to find a new approach which overcomes these difficulties.

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2. GROUP THEORY AND TIME-FREQUENCY ANALYSIS

The approach described here is motivated by group-theoretic generalizations of classical Fourier analysis. The theory of continuous time-frequency distributions is seen as a special case, when the group comprises the real numbers under addition. Properties of continuous time-frequency distributions, [6, 7], are appropriately generalized for discrete groups, and the resulting time-frequency distributions are shown to obey these properties, in a fashion analogous to the continuous time-frequency case. The distributions derived here are compared and contrasted with others proposed in the literature, [1, 2, 3, 4].

As explained in [8], classical Fourier analysis can be extended to a group theoretic setting, in which the continuous-time domain is replaced by a group, the complex exponentials are replaced by the group's irreducible representations, and the continuous-frequency domain is replaced by the dual group. Thus classical Fourier transform may be viewed as a special case of Fourier analysis on a group, where the group is the group of real numbers, $\mathbf{R}$, under addition. The familiar discrete-time Fourier transform, for infinite-length discrete-time signals, may be viewed as performing Fourier analysis on the group of integers $\mathbf{Z}$. The appropriate frequency domain for discrete-time is the dual group of $\mathbf{Z}$, which is the unit circle $\mathbf{T}$ (the set of complex numbers of modulus unity, which form a group under multiplication). The DFT matrix arises from Fourier analysis on the group of integers mod N, $\mathbf{Z}/N$, and the appropriate frequency domain is $\mathbf{Z}/N$ (it is self-dual).

For discrete-time time-frequency distributions, one would expect to look at the group generated by discrete-time shifts and discrete-frequency shifts: this is the approach adopted here. It leads one to investigate the Heisenberg group generated by the integers, [13, 14]. Heisenberg groups have been studied in the mathematical literature, [15, 16, 17]: we draw upon this work, and reinterpret these results in the context of time-frequency analysis.

An advantage of this general setting is that it covers unusual cases where the underlying group is neither continuous-time ($\mathbf{R}$) or discrete-time ($\mathbf{Z}$) but something else. For instance, in the discrete-time, discrete-frequency case, we are dealing with discrete-time N-periodic signals rather than arbitrary discrete-time signals. The underlying group comprises the integers mod N ($\mathbf{Z}/N$): time-frequency analysis

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of these signals is related to the Heisenberg group generated by $\mathbf{Z}/N$, i.e., the group generated by discrete-time and discrete-frequency shifts. This is in contrast with the Heisenberg group generated by discrete-time and continuous frequency shifts. We describe the ambiguity function and Wigner distribution based on the Heisenberg group generated by $\mathbf{Z}/N$, and demonstrate the marginal and Moyal properties.

The following two sections summarize work found in [19].

3. DEFINITION OF A DISCRETE-TIME, DISCRETE-FREQUENCY WIGNER DISTRIBUTION

Define the discrete Fourier transform (DFT) of a discrete, N-periodic signal $x(n)$ to be

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-j2\pi nk/N} x(n), \qquad (1)$$

and the inverse discrete Fourier transform (IDFT) of $X(k)$ to be

$$x(n) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{j2\pi nk/N} X(k). \qquad (2)$$

The non-standard factor $\frac{1}{\sqrt{N}}$ is used for purposes of symmetry. Likewise, the two-dimensional discrete Fourier transform of a discrete function $x(\tau, \nu)$, N-periodic in both arguments, is defined as

$$X(n, k) = \frac{1}{N} \sum_{\tau=0}^{N-1} \sum_{\nu=0}^{N-1} e^{-j2\pi(n\nu+k\tau)/N} x(\tau, \nu), \qquad (3)$$

while the two-dimensional inverse discrete Fourier transform of $X(n, k)$ is defined as

$$x(\tau, \nu) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} e^{j2\pi(n\nu+k\tau)/N} X(n, k). \qquad (4)$$

Now, define the frequency modulation operator, M_ν,

$$(M_\nu x)(n) = e^{j2\pi n\nu/N} x(n), \qquad (5)$$

and the time advance operator, S_τ,

$$(S_\tau x)(n) = x((n+\tau)_N), \qquad (6)$$

using the notation $(a)_N = a \bmod N$. The cross-ambiguity function of two discrete, N-periodic signals x and y, $A_{x,y}(\tau, \nu)$, is defined as the inner product between its first argument $x(n)$, time-shifted and frequency modulated, and its second argument $y(n)$:

$$A_{x,y}(\tau, \nu) = (e^{j\frac{\pi(\nu\tau)_N}{N}} M_\nu S_\tau x, y) =$$

$$\sum_{l=0}^{N-1} e^{j\frac{\pi(\nu\tau)_N}{N}} e^{j\frac{2\pi\nu l}{N}} x((l+\tau)_N) y^*(l). \qquad (7)$$

This expression for the cross-ambiguity function is the discrete realization of the "symmetric" cross-ambiguity function defined in [18]. The phase factor $e^{j\frac{\pi(\nu\tau)_N}{N}}$ is present to preserve commutativity between the time advance and frequency modulation operators.

The new discrete cross-Wigner distribution, $W_{x,y}(n, k)$, which we call WS, is the two-dimensional Fourier transform of the cross-ambiguity function, $A_{x,y}(\tau, \nu)$,

$$W_{x,y}(n, k) =$$

$$\frac{1}{N} \sum_{\tau=0}^{N-1} \sum_{\nu=0}^{N-1} \sum_{l=0}^{N-1} e^{-j\frac{2\pi(n\nu+k\tau)}{N}} e^{j\frac{\pi(\nu\tau)_N}{N}} e^{j\frac{2\pi\nu l}{N}} x((l+\tau)_N) y^*(l). \qquad (8)$$

4. PROPERTIES OF THE NEW DISCRETE WIGNER DISTRIBUTION

We can demonstrate that the time-marginal property holds for WS by summing the auto-Wigner distribution of x, $W_{x,x}(n, k)$, over all n for fixed k, to obtain

$$\sum_{n=0}^{N-1} W_{x,x}(n, k)$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{\tau=0}^{N-1} \sum_{\nu=0}^{N-1} e^{-j\frac{2\pi(n\nu+k\tau)}{N}} A_{x,x}(\tau, \nu),$$

$$= \frac{1}{N} \sum_{\tau=0}^{N-1} \sum_{l=0}^{N-1} \left[e^{-j\frac{2\pi k\tau}{N}} x((l+\tau)_N) x^*(l) \right]$$

$$\left[\sum_{\nu=0}^{N-1} e^{j\frac{\pi(\nu\tau)_N}{N}} e^{j\frac{2\pi\nu l}{N}} \sum_{n=0}^{N-1} e^{-j\frac{2\pi n\nu}{N}} \right],$$

$$= \sum_{\tau=0}^{N-1} \sum_{l=0}^{N-1} e^{-j\frac{2\pi k\tau}{N}} x((l+\tau)_N) x^*(l),$$

$$= \sum_{l=0}^{N-1} x^*(l) \sum_{\tau=0}^{N-1} e^{-j\frac{2\pi k\tau}{N}} x((l+\tau)_N),$$

$$= \sqrt{N} \sum_{l=0}^{N-1} x^*(l) e^{j\frac{2\pi kl}{N}} X(k).$$

We thus derive the time marginal property:

$$\sum_{n=0}^{N-1} W_{x,x}(n, k) = N|X(k)|^2. \qquad (9)$$

WS also satisfies the following frequency marginal property:

$$\sum_{k=0}^{N-1} W_{x,x}(n, k) = N|x(n)|^2. \qquad (10)$$

The two-dimensional inner product of the cross ambiguity functions A_{x_1,y_1} and A_{x_2,y_2} is

$$(A_{x_1,y_1}, A_{x_2,y_2}) =$$

$$\sum_{\tau=0}^{N-1}\sum_{\nu=0}^{N-1}(e^{j\frac{\pi(\nu\tau)_N}{N}}M_\nu S_\tau x_1, y_1)(e^{j\frac{\pi(\nu\tau)_N}{N}}M_\nu S_\tau x_2, y_2)^*,$$

$$= \sum_{\tau=0}^{N-1}\sum_{\nu=0}^{N-1}\left[\sum_{l=0}^{N-1} e^{j\frac{\pi(\nu\tau)_N}{N}} e^{j\frac{2\pi\nu l}{N}} x_1((l+\tau)_N y_1^*(l)\right]$$

$$\left[\sum_{m=0}^{N-1} e^{j\frac{\pi(\nu\tau)_N}{N}} e^{j\frac{2\pi\nu m}{N}} x_2((m+\tau)_N y_2^*(m)\right]^*,$$

$$= N\sum_{\tau=0}^{N-1}\sum_{l=0}^{N-1} x_1((l+\tau)_N) y_1^*(l) x_2^*((l+\tau)_N) y_2(l),$$

$$= N\sum_{l=0}^{N-1} y_1^*(l) y_2(l) \sum_{\tau=0}^{N-1} x_2^*((l+\tau)_N) x_1((l+\tau)_N),$$

$$= N(y_1, y_2)^*(x_1, x_2),$$

Thus,

$$(W_{x_1,y_1}, W_{x_2,y_2}) = (A_{x_1,y_1}, A_{x_2,y_2}) = N(x_1,x_2)(y_1,y_2)^*, \tag{11}$$

so WS satisfies Moyal's formula.

We also note that the auto-WS Wigner distribution is real and that the convolution of the WS Wigner distribution of a signal with the WS Wigner distribution of a window is a spectrogram.

5. EXAMPLE AND COMPARISONS

We evaluate the new Wigner distribution (WS) for a test signal consisting of a cosine ($cos(2\pi(0.4)t)$) and a chirp ($e^{-2\pi j*48*t^2}$) spaced in time: in *MATLAB* notation,

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x=[cos(4*pi/5*[0:63]');zeros(48,1);...
exp(-96*j*pi*[0:1/128:5/8-1/128]'.^2)]
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We also compare this result with the computed distributions specified by Claasen and Mecklenbräuker (WCM) and by Peyrin and Prost (WPP). Each of WS, WCM, and WPP is illustrated in Figure 1 for the given test signal. As shown in the figure, the distribution described by WS does not suffer from aliasing for either the cosine component or the chirp component, even though each of these signal components has frequencies greater than $\frac{f_s}{4}$. WS can thus be characterized as a full-band time-frequency representation. WCM and WPP cannot be characterized in this way, as aliasing occurs for the given test signal. Another feature shown in Figure 1 is the unusual interference properties of WS. Strong interference terms appear only where there is time support and frequency support in the signal. This is quite different from the interference shown for WCM and WPP, where terms appear halfway in between signal components in time and frequency.

There are some interesting relationships among these 3 distributions. Let y_N be a given length-N signal and y_{2N} be the signal obtained by padding y_N to length-$2N$. Then WCM for y_N is equivalent to the result obtained by decimating WPP for y_{2N} by 2 in both time and frequency. Also, the magnitude of the ambiguity function corresponding to WCM for y_N is equivalent to the result obtained by decimating the magnitude of the ambiguity function corresponding to WS by 2 in both time lag and frequency lag. However, there is no simple relationship between either WS and WPP or their corresponding ambiguity functions. We thus interpret WPP as an interpolation of WCM in the time-frequency plane, and we interpret WS as an interpolation of WCM in the ambiguity plane.

Each of the discrete-time, discrete-frequency Wigner distributions proposed in the literature has various advantages and disadvantages for use in studying signals in the time-frequency plane. The new distribution derived here also demonstrates qualities which merit its use in discrete-time, discrete-frequency analysis of signals.

6. REFERENCES

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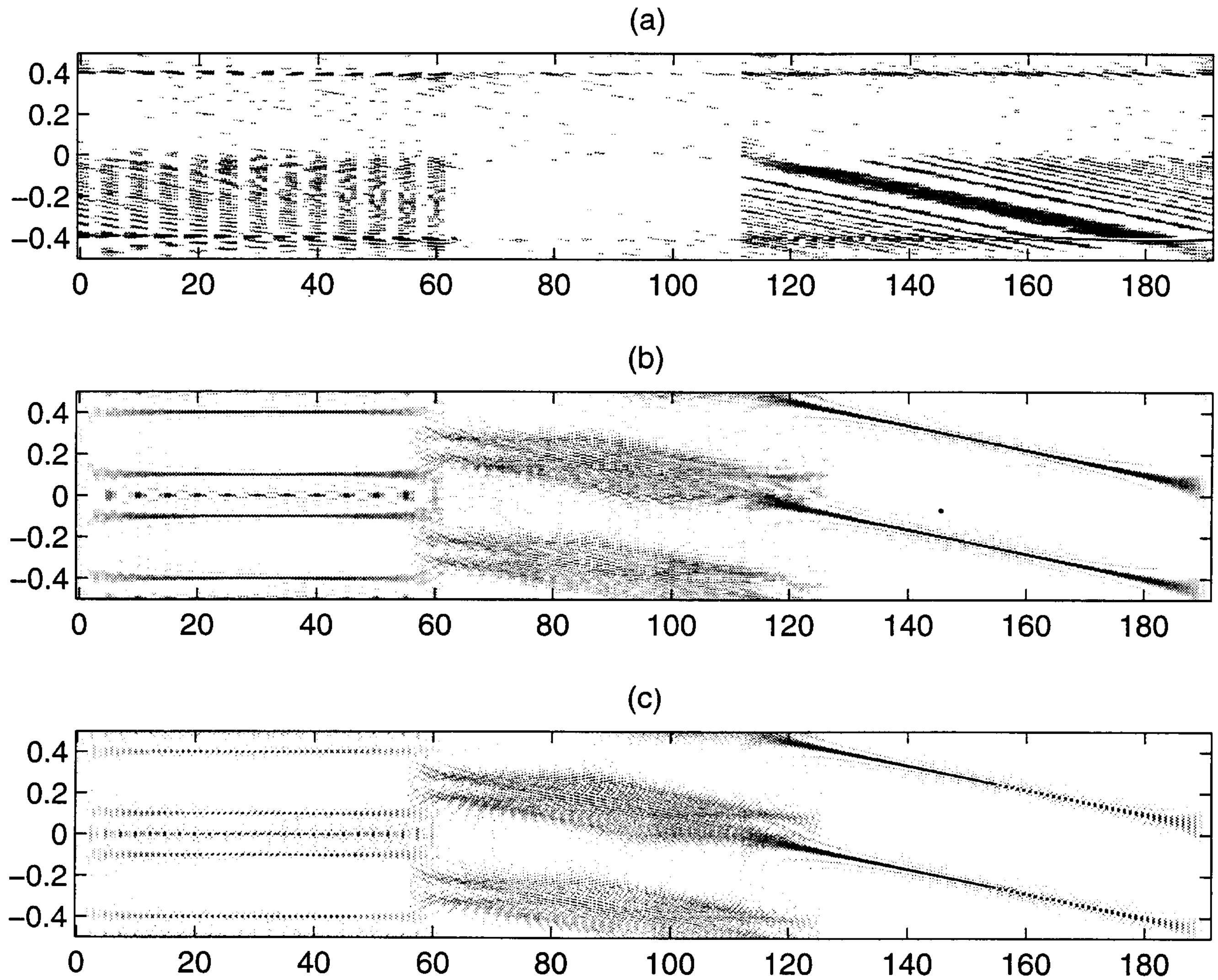

Figure 1: Comparison of Time-Frequency Distributions: a) Shenoy Wigner Distribution, WS b) Claasen, Mecklenbräuker Wigner Distribution, WCM c) Peyrin, Prost Wigner Distribution, WPP (using a padded signal). The horizontal axis denotes time. The vertical axis denotes normalized frequency.

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