DISPLACEMENT-COVARIANT TIME-FREQUENCY ENERGY DISTRIBUTIONS*

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Abstract—We present a theory of quadratic time-frequency (TF) energy distributions that satisfy a covariance property and generalized marginal properties. The theory coincides with the characteristic function method of Cohen and Baraniuk in the special case of "conjugate operators."

1 INTRODUCTION AND OUTLINE

Important classes of quadratic time-frequency representations (QTFRs), such as Cohen's class[1] and the affine, hyperbolic, and power classes [1]-[8], are special cases within a general theory of *displacement-covariant QTFRs* [9]. This theory (briefly reviewed in Section 2) is based on the concept of *time-frequency displacement operators* (DOs).

In Section 3, we shall consider the important *separable case* where a DO can be decomposed into two "partial DOs" (PDOs). Section 4 defines *marginal properties* associated to the PDOs and derives constraints on the QTFR kernels. Section 5 shows that, for "conjugate" PDOs, our theory coincides with the characteristic function method of [10, 11].

2 DISPLACEMENT-COVARIANT QTFRs

Time-Frequency Displacement Operators. A DO is a family of unitary, linear operators $\mathbf{D}_\theta$ defined on a linear space $\mathcal{X} \subseteq \mathcal{L}_2(\mathbb{R})$ of finite-energy signals $x(t)$, and indexed by the 2D "displacement parameter" $\theta = (\alpha, \beta) \in \mathcal{D}$ with $\mathcal{D} \subseteq \mathbb{R}^2$. By definition, $\mathbf{D}_\theta$ obeys a *composition law*

$$\mathbf{D}_{\theta_2}\mathbf{D}_{\theta_1} = e^{j\sigma(\theta_1,\theta_2)}\, \mathbf{D}_{\theta_1 \circ \theta_2} \tag{1}$$

where $\circ$ is a binary operation such that $\mathcal{D}$ and $\circ$ form a group[2] with identity element θ_0 and inverse element θ^{-1}. The TF displacements produced by a DO are described by its *displacement function* (DF) $d(z,\theta)$: if a signal $x(t)$ is localized about a TF point $z = (t, f)$, then $(\mathbf{D}_\theta\, x)(t)$ is localized about some other TF point $z' = (t', f')$ given by

$$z' = d(z, \theta)\,,$$

which is short for $t' = d_1(t, f; \alpha, \beta)$, $f' = d_2(t, f; \alpha, \beta)$. The DF's construction is discussed in [9]. The DF is assumed to be an invertible, area-preserving mapping of $\mathcal{Z}$ onto $\mathcal{Z}$ (where $\mathcal{Z} \subseteq \mathbb{R}^2$ denotes the set of TF points $z = (t, f)$), and to obey the composition law (cf. (1))

$$d\big(d(z, \theta_1), \theta_2\big) = d(z,\, \theta_1 \circ \theta_2)\,. \tag{2}$$

The *parameter function* $p(z', z)$ of $\mathbf{D}_\theta$ yields the displacement parameter θ that maps z into z',

$$z' = d(z, \theta) \quad \Leftrightarrow \quad \theta = p(z', z)\,,$$

which is short for $\alpha = p_1(t', f'; t, f)$, $\beta = p_2(t', f'; t, f)$.

Two Examples. The *TF-shift operator* $\mathbf{S}_{\tau,\nu}$, defined as $(\mathbf{S}_{\tau,\nu}x)(t) = x(t-\tau)\, e^{j2\pi\nu t}$, is a DO with composition law (1) $\mathbf{S}_{\tau_2,\nu_2}\mathbf{S}_{\tau_1,\nu_1} = e^{-j2\pi\nu_1\tau_2}\, \mathbf{S}_{\tau_1+\tau_2,\, \nu_1+\nu_2}$, DF $t' = d_1(t, f; \tau, \nu) = t + \tau$, $f' = d_2(t, f; \tau, \nu) = f + \nu$, and parameter function $\tau = p_1(t', f'; t, f) = t' - t$, $\nu = p_2(t', f'; t, f) = f' - f$. Another DO is the *time-shift/TF-scaling operator* $\mathbf{C}_{a,\tau}$ defined as $(\mathbf{C}_{a,\tau}\, x)(t) = \sqrt{a}\ x\big(a(t-\tau)\big)$ $(a > 0)$, with $\mathbf{C}_{a_2,\tau_2}\mathbf{C}_{a_1,\tau_1} = \mathbf{C}_{a_1 a_2,\, \tau_1/a_2+\tau_2}$, DF $t' = d_1(t, f; a, \tau) = t/a + \tau$, $f' = d_2(t, f; a, \tau) = af$, and parameter function $a = p_1(t', f'; t, f) = f'/f$, $\tau = p_2(t', f'; t, f) = t' - tf/f'$.

Displacement-Covariant QTFRs. A QTFR $T_x(t, f) = T_x(z)$ is called *covariant to a DO* $\mathbf{D}_\theta$ if

$$T_{\mathbf{D}_\theta x}(z) = T_x(\tilde{z}) \qquad \text{with } \tilde{z} = d(z, \theta^{-1})\,. \tag{3}$$

It can be shown [9] that all QTFRs satisfying the covariance property (3) are given by the 2D inner product[3]

$$T_x(z) = \int_{t_1}\!\int_{t_2} x(t_1)\, x^*(t_2)\, \big(\mathbf{D}^{\otimes}_{p(z,z_0)} h\big)^*(t_1, t_2)\, dt_1 dt_2 \tag{4}$$

$$= \int_{f_1}\!\int_{f_2} X(f_1)\, X^*(f_2)\, \big(\hat{\mathbf{D}}^{\otimes}_{p(z,z_0)} H\big)^*(f_1, f_2)\, df_1 df_2 \tag{5}$$

where $h(t_1, t_2)$ is a 2D "kernel" (independent of $x(t)$), $z_0 \in \mathcal{Z}$ is a fixed reference TF point, $\mathbf{D}^{\otimes}_\theta$ is the outer product of $\mathbf{D}_\theta$ by itself[4], $X(f) = \mathcal{F}_{t\to f}\, x(t)$, $\hat{\mathbf{D}}_\theta = \mathcal{F}\mathbf{D}_\theta\mathcal{F}^{-1}$, and $H(f_1, f_2) = \mathcal{F}_{t_1\to f_1}\mathcal{F}_{t_2\to -f_2}\, h(t_1, t_2)$. Conversely, all QTFRs (4),(5) are covariant to $\mathbf{D}_\theta$. We note that (4) can be written as the quadratic form

$$T_x(z) = \big\langle x,\, \mathbf{H}^D_z x\big\rangle \quad \text{with } \mathbf{H}^D_z = \mathbf{D}_{p(z,z_0)}\mathbf{H}\,\mathbf{D}^{-1}_{p(z,z_0)}\,, \tag{6}$$

where $\mathbf{H}$ is the linear operator whose kernel is $h(t_1, t_2)$, i.e. $(\mathbf{H}\, x)(t) = \int_{t'} h(t, t')\, x(t')\, dt'$, and $\langle x, y\rangle = \int_t x(t)\, y^*(t)\, dt$.

Examples. For $\mathbf{D}_\theta = \mathbf{S}_{\tau,\nu}$ and $z_0 = (0, 0)$, (3) becomes the TF-shift covariance $T_{\mathbf{S}_{\tau,\nu}x}(t, f) = T_x(t-\tau, f-\nu)$ and (4) becomes *Cohen's class* [1]-[3]

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[1] Short for *Cohen's class with signal-independent kernels*.

[2] The group axioms are (i) $\theta_1 \circ \theta_2 \in \mathcal{D}$ for $\theta_1, \theta_2 \in \mathcal{D}$, (ii) $\theta_1 \circ (\theta_2 \circ \theta_3) = (\theta_1 \circ \theta_2) \circ \theta_3$, (iii) $\theta \circ \theta_0 = \theta_0 \circ \theta = \theta$, and (iv) $\theta^{-1} \circ \theta = \theta \circ \theta^{-1} = \theta_0$.

[3] Integrals are over the functions' support.

[4] $\mathbf{D}^{\otimes}_\theta$ acts on a 2D function $y(t_1, t_2)$ as $\big(\mathbf{D}^{\otimes}_\theta y\big)(t_1, t_2) = \int_{t_1'}\int_{t_2'} D_\theta(t_1, t_1')\, D^*_\theta(t_2, t_2')\, y(t_1', t_2')\, dt_1' dt_2'$, where $D_\theta(t, t')$ is the kernel of $\mathbf{D}_\theta$. For example, $(\mathbf{S}^{\otimes}_{\tau,\nu}\, y)(t_1, t_2) = y(t_1-\tau, t_2-\tau)\, e^{j2\pi\nu(t_1-t_2)}$ and $(\mathbf{C}^{\otimes}_{a,\tau}\, y)(t_1, t_2) = a\, y\big(a(t_1-\tau), a(t_2-\tau)\big)$.

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$$T_x(t,f) = \int_{t_1}\int_{t_2} x(t_1)\, x^*(t_2)\, h^*(t_1-t, t_2-t)\, e^{-j2\pi f(t_1-t_2)} dt_1 dt_2. \quad (7)$$

For $\mathbf{D}_\theta = \mathbf{C}_{a,\tau}$ and $z_0 = (0, f_0)$ (with fixed $f_0 > 0$), (3) becomes the time-shift/TF-scaling covariance $T_{\mathbf{C}_{a,\tau}x}(t,f) = T_x\big(a(t-\tau), f/a\big)$ and (4) becomes the *affine class* [4, 5]

$$T_x(t,f) = \frac{f}{f_0}\int_{t_1}\int_{t_2} x(t_1)\, x^*(t_2)\, h^*\Big(\frac{f}{f_0}(t_1-t), \frac{f}{f_0}(t_2-t)\Big) dt_1 dt_2 \quad (8)$$

for $f > 0$. Further special cases of (4)-(6) are the *hyperbolic class* and the *power classes* [6]-[9].

3 THE SEPARABLE CASE

The next theorem (obtained from (1), (2)) considers a *separable* DO that can be decomposed into two "partial DOs."

Theorem 1. Let $\mathbf{D}_\theta$ with $\theta = (\alpha, \beta)$, $\mathcal{D} = \mathcal{A} \times \mathcal{B}$ be a DO with identity parameter $\theta_0 = (\alpha_0, \beta_0)$, and define $\theta_\alpha = (\alpha, \beta_0)$ and $\theta_\beta = (\alpha_0, \beta)$. If

$$\theta_\alpha \circ \theta_\beta = \theta, \qquad \theta_{\alpha_1} \circ \theta_{\alpha_2} = \theta_{\alpha_{12}}, \qquad \theta_{\beta_1} \circ \theta_{\beta_2} = \theta_{\beta_{12}} \quad (9)$$

with $\alpha_{12} = \alpha_1 \bullet \alpha_2$ and $\beta_{12} = \beta_1 * \beta_2$, where $\bullet$ and $*$ are commutative operations, then the following results hold:[5]

(i) The DO $\mathbf{D}_\theta$ can be decomposed as

$$\mathbf{D}_\theta = e^{-j\sigma(\theta_\alpha, \theta_\beta)}\, \mathbf{B}_\beta \mathbf{A}_\alpha$$

with the *partial DOs* (PDOs) $\mathbf{A}_\alpha = \mathbf{D}_{\theta_\alpha}$ and $\mathbf{B}_\beta = \mathbf{D}_{\theta_\beta}$.

(ii) The PDO $\mathbf{A}_\alpha$ is a family of linear operators indexed by the 1D displacement parameter $\alpha \in \mathcal{A}$ with $\mathcal{A} \subseteq \mathbb{R}$. $\mathbf{A}_\alpha$ is unitary on $\mathcal{X}$ and satisfies the composition law

$$\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = e^{j\sigma(\theta_{\alpha_1}, \theta_{\alpha_2})}\, \mathbf{A}_{\alpha_1 \bullet \alpha_2},$$

where $\mathcal{A}$ and $\bullet$ form a commutative group with identity element α_0. Analogous results hold for the PDO $\mathbf{B}_\beta$.

(iii) The DF of $\mathbf{D}_\theta$ can be decomposed as $d(z,\theta) = d^B\big(d^A(z,\alpha), \beta\big)$ with the *partial DFs* $d^A(z,\alpha) = d(z, \theta_\alpha)$ and $d^B(z,\beta) = d(z,\theta_\beta)$.

In the following, we assume $\sigma(\theta_{\alpha_1}, \theta_{\alpha_2}) = \sigma(\theta_{\beta_1}, \theta_{\beta_2}) \equiv 0$ so that $\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1 \bullet \alpha_2}$ and $\mathbf{B}_{\beta_2}\mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1 * \beta_2}$.

Eigenvalues and Eigenfunctions [10, 12]. The *eigenvalues* $\lambda^A_{\alpha,\tilde{\alpha}}$ and *eigenfunctions* $u^A_{\tilde{\alpha}}(t)$ of $\mathbf{A}_\alpha$ are defined by

$$\big(\mathbf{A}_\alpha u^A_{\tilde{\alpha}}\big)(t) = \lambda^A_{\alpha,\tilde{\alpha}}\, u^A_{\tilde{\alpha}}(t); \quad (10)$$

they are indexed by a "dual parameter" $\tilde{\alpha} \in \tilde{\mathcal{A}}$ with $\tilde{\mathcal{A}} \subseteq \mathbb{R}$. The composition law $\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1 \bullet \alpha_2}$ implies $\lambda^A_{\alpha_1 \bullet \alpha_2, \tilde{\alpha}} = \lambda^A_{\alpha_1,\tilde{\alpha}}\, \lambda^A_{\alpha_2,\tilde{\alpha}}$, and the unitarity of $\mathbf{A}_\alpha$ implies $|\lambda^A_{\alpha,\tilde{\alpha}}| \equiv 1$. It follows [13] that $\tilde{\alpha}$ belongs to a commutative "dual" group $(\tilde{\mathcal{A}}, \tilde{\bullet})$ and that there is $\lambda^A_{\alpha, \tilde{\alpha}_1 \tilde{\bullet} \tilde{\alpha}_2} = \lambda^A_{\alpha,\tilde{\alpha}_1}\, \lambda^A_{\alpha,\tilde{\alpha}_2}$. These relations show that the eigenvalues must be of the form

$$\lambda^A_{\alpha,\tilde{\alpha}} = e^{j2\pi\, \mu_A(\alpha)\, \tilde{\mu}_A(\tilde{\alpha})}, \quad (11)$$

where $\mu_A(\alpha_1 \bullet \alpha_2) = \mu_A(\alpha_1) + \mu_A(\alpha_2)$, $\mu_A(\alpha_0) = 0$, $\mu_A(\alpha^{-1}) = -\mu_A(\alpha)$, and $\tilde{\mu}_A(\tilde{\alpha}_1 \tilde{\bullet} \tilde{\alpha}_2) = \tilde{\mu}_A(\tilde{\alpha}_1) + \tilde{\mu}_A(\tilde{\alpha}_2)$, $\tilde{\mu}_A(\tilde{\alpha}_0) = 0$, $\tilde{\mu}_A(\tilde{\alpha}^{-1}) = -\tilde{\mu}_A(\tilde{\alpha})$. This implies $\lambda^A_{\alpha_0,\tilde{\alpha}} = \lambda^A_{\alpha,\tilde{\alpha}_0} = 1$ and $\lambda^A_{\alpha^{-1},\tilde{\alpha}} = \lambda^A_{\alpha,\tilde{\alpha}^{-1}} = \lambda^{A*}_{\alpha,\tilde{\alpha}}$. Analogous results hold for $\mathbf{B}_\beta$.

[5] Analogous results hold if $\theta_\beta \circ \theta_\alpha = \theta$.

A-Fourier Transform. Assuming suitable normalization of the eigenfunctions $u^A_{\tilde{\alpha}}(t)$, it can be shown [10, 12] that any $x(t) \in \mathcal{X}$ can be expanded into the $u^A_{\tilde{\alpha}}(t)$ as

$$x(t) = \int_{\tilde{\mathcal{A}}} X_A(\tilde{\alpha})\, u^A_{\tilde{\alpha}}(t)\, |\tilde{\mu}'_A(\tilde{\alpha})|\, d\tilde{\alpha} = (\mathcal{F}_A^{-1} X_A)(t), \quad (12)$$

with the **A**-*Fourier transform* (**A**-FT) [10, 12]

$$X_A(\tilde{\alpha}) = \big\langle x, u^A_{\tilde{\alpha}} \big\rangle = \int_t x(t)\, u^{A*}_{\tilde{\alpha}}(t)\, dt = (\mathcal{F}_A\, x)(\tilde{\alpha}). \quad (13)$$

$|X_A(\tilde{\alpha})|^2$ is an *energy density* since $\int_{\tilde{\mathcal{A}}} |X_A(\tilde{\alpha})|^2 |\tilde{\mu}'_A(\tilde{\alpha})|\, d\tilde{\alpha} = \int_t |x(t)|^2 dt = \|x\|^2$. With (10), (12), and (13) we easily show

$$(\mathbf{A}_\alpha\, x)(t) = \int_{\tilde{\mathcal{A}}} \lambda^A_{\alpha,\tilde{\alpha}} \big\langle x, u^A_{\tilde{\alpha}} \big\rangle\, u^A_{\tilde{\alpha}}(t)\, |\tilde{\mu}'_A(\tilde{\alpha})|\, d\tilde{\alpha}. \quad (14)$$

Displacement Curves. The TF displacements produced by a PDO $\mathbf{A}_\alpha$ are described by the partial DF $z' = d^A(z,\alpha)$ (see Theorem 1), which is short for $t' = d^A_1(t,f;\alpha)$, $f' = d^A_2(t,f;\alpha)$. For given z, the set of all $z' = d^A(z,\alpha)$ obtained by varying α is a curve $\mathcal{C}^A_z \in \mathcal{Z}$ that passes through z. This curve will be called a *displacement curve* (DC) of the PDO $\mathbf{A}_\alpha$. The eigenequation (10) implies that $\mathbf{A}_\alpha$ does not cause a TF displacement of $u^A_{\tilde{\alpha}}(t)$. *Hence, $u^A_{\tilde{\alpha}}(t)$ must be TF-localized along a DC $\mathcal{C}^A_z$, where z is related to the eigenfunction index $\tilde{\alpha}$.* Two cases will be considered:

Case 1. The eigenfunction can be written as

$$u^A_{\tilde{\alpha}}(t) = r^A_{\tilde{\alpha}}(t)\, e^{j2\pi [b_A(\tilde{\alpha})\, \phi_A(t) + \psi_A(t)]}, \quad (15)$$

where $b_A(\tilde{\alpha})$ and $\phi_A(t)$ are one-to-one functions and $r^A_{\tilde{\alpha}}(t) = \sqrt{|b'_A(\tilde{\alpha})\, \phi'_A(t)/\tilde{\mu}'_A(\tilde{\alpha})|}$ in order to be consistent with (12), (13). Here, the DC $\mathcal{C}^A_z$ is postulated to coincide with the *instantaneous frequency*

$$\nu^A_{\tilde{\alpha}}(t) = b_A(\tilde{\alpha})\, \phi'_A(t) + \psi'_A(t) \quad (16)$$

of $u^A_{\tilde{\alpha}}(t)$, where $z = (t,f)$ in $\mathcal{C}^A_z$ is related to $\tilde{\alpha}$ in that z lies on the instantaneous-frequency curve, i.e. $f = \nu^A_{\tilde{\alpha}}(t)$.

Case 2. The Fourier transform of $u^A_{\tilde{\alpha}}(t)$ can be written as

$$U^A_{\tilde{\alpha}}(f) = R^A_{\tilde{\alpha}}(f)\, e^{-j2\pi [b_A(\tilde{\alpha})\, \Phi_A(f) + \Psi_A(f)]}, \quad (17)$$

where $b_A(\tilde{\alpha})$ and $\Phi_A(f)$ are one-to-one functions and $R^A_{\tilde{\alpha}}(f) = \sqrt{|b'_A(\tilde{\alpha})\, \Phi'_A(f)/\tilde{\mu}'_A(\tilde{\alpha})|}$. Here, $\mathcal{C}^A_z$ is postulated to coincide with the *group delay*

$$\tau^A_{\tilde{\alpha}}(f) = b_A(\tilde{\alpha})\, \Phi'_A(f) + \Psi'_A(f) \quad (18)$$

of $u^A_{\tilde{\alpha}}(t)$, where $z = (t,f)$ in $\mathcal{C}^A_z$ is related to $\tilde{\alpha}$ as $t = \tau^A_{\tilde{\alpha}}(f)$.

Since in both cases the DC $\mathcal{C}^A_z$ is really parameterized by $\tilde{\alpha}$, we shall henceforth write $\mathcal{C}^A_{\tilde{\alpha}}$.

Examples. The DOs $\mathbf{S}_{\tau,\nu}$ and $\mathbf{C}_{a,\tau}$ are both separable. We have $\mathbf{S}_{\tau,\nu} = \mathbf{F}_\nu \mathbf{T}_\tau$ and $\mathbf{C}_{a,\tau} = \mathbf{T}_\tau \mathbf{L}_a$ with the time-shift operator $\mathbf{T}_\tau$, frequency-shift operator $\mathbf{F}_\nu$, and TF-scaling operator $\mathbf{L}_a$ defined by $(\mathbf{T}_\tau\, x)(t) = x(t-\tau)$, $(\mathbf{F}_\nu\, x)(t) = x(t)\, e^{j2\pi\nu t}$, and $(\mathbf{L}_a\, x)(t) = \sqrt{a}\, x(at)$ $(a > 0)$.

$\mathbf{T}_\tau$ is a "case-1 PDO" with $(\mathcal{A}, \bullet) = (\tilde{\mathcal{A}}, \tilde{\bullet}) = (\mathbb{R}, +)$, $\lambda^T_{\tau,f} = e^{-j2\pi\tau f}$, $u^T_f(t) = e^{j2\pi f t}$, $\tilde{\tau} = f$, $\mu_T(\tau) = -\tau$, $\tilde{\mu}_T(f) = f$, $b_T(f) = f$, $\phi_T(t) = t$, and $\psi_T(t) \equiv 0$. The DC $\mathcal{C}^T_{t,f}$: $(t', f') = (t+\tau, f)$ coincides with the instantaneous frequency $\nu^T_f(t) = f$, and the **T**-FT is the Fourier transform, $X_T(f) = \int_t x(t)\, e^{-j2\pi f t} dt = X(f)$.

$\mathbf{F}_\nu$ is a "case-2 PDO" with $(\mathcal{A},\bullet) = (\tilde{\mathcal{A}},\tilde{\bullet}) = (\mathbb{R},+)$, $\lambda^F_{\nu,t} = e^{j2\pi\nu t}$, $U^F_t(f) = e^{-j2\pi tf}$, $\tilde{\nu}=t$, $\mu_F(\nu)=\nu$, $\tilde{\mu}_F(t)=t$, $b_F(t)=t$, $\Phi_F(f)=f$, and $\Psi_F(f)\equiv 0$. The DC $\mathcal{C}^F_{t,f}$: $(t',f')=(t,f+\nu)$ coincides with the group delay $\tau^F_t(f)=t$, and the F-FT is the identity transform, $X_F(t)=x(t)$.

$\mathbf{L}_a$ (defined for analytic signals) is a "case-2 PDO" with $(\mathcal{A},\bullet) = (\mathbb{R}_+,\cdot)$, $(\tilde{\mathcal{A}},\tilde{\bullet}) = (\mathbb{R},+)$, $\lambda^L_{a,c} = e^{j2\pi c\ln a}$, $U^L_c(f) = e^{-j2\pi c\ln(f/f_r)}/\sqrt{f}$ for $f>0$ (with fixed $f_r>0$), $\tilde{a}=c$, $\mu_L(a)=\ln a$, $\tilde{\mu}_L(c)=c$, $b_L(c)=c$, $\Phi_L(f)=\ln(f/f_r)$, and $\Psi_L(f)\equiv 0$. The DC $\mathcal{C}^L_{t,f}$: $(t',f')=(at,f/a)$ coincides with the group delay $\tau^L_c(f)=c/f$, and the L-FT is the Mellin transform [6, 14, 11] $X_L(c)=\int_0^\infty X(f)\,e^{j2\pi c\ln(f/f_r)}df/\sqrt{f}$.

Furthermore, also the DOs underlying the hyperbolic and power classes [6]-[9] are separable.

4 MARGINAL PROPERTIES

We now consider a separable DO $\mathbf{D}_\theta = e^{-j\sigma(\theta_\alpha,\theta_\beta)}\,\mathbf{B}_\beta\mathbf{A}_\alpha$ where $\mathbf{A}_\alpha$ is a case-1 PDO and $\mathbf{B}_\beta$ is a case-2 PDO (analogous results hold if $\mathbf{A}_\alpha$ is case 2 and $\mathbf{B}_\beta$ is case 1).

Marginal Properties and Kernel Constraints. The *marginal property* associated to the PDO $\mathbf{A}_\alpha$ states that integration of a QTFR $T_x(t,f)$ over the DC $\mathcal{C}^A_{\tilde{\alpha}}$ (the TF locus of $u^A_{\tilde{\alpha}}(t)$) yields the energy density $|X_A(\tilde{\alpha})|^2 = \left|\langle x, u^A_{\tilde{\alpha}}\rangle\right|^2$:

$$\int_t T_x\big(t,\nu^A_{\tilde{\alpha}}(t)\big)\,\big[r^A_{\tilde{\alpha}}(t)\big]^2\,dt = |X_A(\tilde{\alpha})|^2\,. \qquad (19)$$

Similarly, the marginal property associated to $\mathbf{B}_\beta$ reads

$$\int_f T_x\big(\tau^B_{\tilde{\beta}}(f),f\big)\,\big[R^B_{\tilde{\beta}}(f)\big]^2\,df = |X_B(\tilde{\beta})|^2\,. \qquad (20)$$

It can be shown that a QTFR $T_x(t,f)$ covariant to the DO $\mathbf{D}_\theta$ satisfies the marginal property (19) if and only if its kernel $h(t_1,t_2)$ (cf. (4)) satisfies the constraint

$$\int_t \big(\mathbf{D}^{\otimes}_{p(z(t),z_0)}h\big)(t_1,t_2)\,\big[r^A_{\tilde{\alpha}}(t)\big]^2\,dt = u^A_{\tilde{\alpha}}(t_1)\,u^{A*}_{\tilde{\alpha}}(t_2) \qquad (21)$$

with $z(t) = \big(t,\nu^A_{\tilde{\alpha}}(t)\big)$. Similarly, (20) holds if and only if

$$\int_f \big(\hat{\mathbf{D}}^{\otimes}_{p(z(f),z_0)}H\big)(f_1,f_2)\,\big[R^B_{\tilde{\beta}}(f)\big]^2 df = U^B_{\tilde{\beta}}(f_1)\,U^{B*}_{\tilde{\beta}}(f_2) \qquad (22)$$

with $z(f) = \big(\tau^B_{\tilde{\beta}}(f),f\big)$, where $H(f_1,f_2)$ is the kernel in (5).

Examples. From (19), (20), the marginal properties associated to $\mathbf{T}_\tau$, $\mathbf{F}_\nu$, and $\mathbf{L}_a$ follow as $\int_t T_x(t,f)\,dt = |X(f)|^2$, $\int_f T_x(t,f)\,df = |x(t)|^2$, and $\int_f T_x(c/f,f)\,df/f = |X_L(c)|^2$, respectively. For Cohen's class (7), the constraints for the $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$ marginal properties follow from (21), (22), after simplification, as $\int_t h(t_1-t,t_2-t)\,dt = 1\ \forall t_1,t_2$ and $\int_f H(f_1-f,f_2-f)\,df = 1\ \forall f_1,f_2$, respectively. For the affine class (8), the constraints for the $\mathbf{L}_a$ and $\mathbf{T}_\tau$ marginal properties follow as $f_0\int_0^\infty H(f_0f_1/f,f_0f_2/f)\,e^{-j2\pi(f_1-f_2)c/f}\,df/f^2 = e^{-j2\pi c\ln(f_1/f_2)}/\sqrt{f_1f_2}$ and $(f/f_0)\int_t h\big(f(t_1-t)/f_0, f(t_2-t)/f_0\big)\,dt = e^{j2\pi f(t_1-t_2)}$, respectively.

Localization Function. We now assume that the DCs $\mathcal{C}^A_{\tilde{\alpha}}$, $\mathcal{C}^B_{\tilde{\beta}}$ corresponding to a dual parameter pair $\tilde{\theta} = (\tilde{\alpha},\tilde{\beta})$ intersect in a unique TF point

$$z = l(\tilde{\theta})\,,$$

which is short for $t = l_1(\tilde{\alpha},\tilde{\beta})$, $f = l_2(\tilde{\alpha},\tilde{\beta})$. We shall call $l(\tilde{\theta})$ the *localization function* (LF) of the separable DO $\mathbf{D}_\theta$. The LF is constructed by solving the system of equations $\nu^A_{\tilde{\alpha}}(t) = f$, $\tau^B_{\tilde{\beta}}(f) = t$ for $(t,f) = z$ [12]. We assume that, to any $z \in \mathcal{Z}$, there exists a unique $\tilde{\theta} = (\tilde{\alpha},\tilde{\beta})$ such that $z = l(\tilde{\theta})$. Hence, $\tilde{\theta} = l^{-1}(z)$ with the inverse LF $l^{-1}(z)$. The marginal properties (19), (20) can now be written as

$$\int_{\tilde{\mathcal{B}}} T_x\big(l(\tilde{\theta})\big)\,n_1(\tilde{\theta})\,d\tilde{\beta} = |X_A(\tilde{\alpha})|^2 \qquad (23)$$

$$\int_{\tilde{\mathcal{A}}} T_x\big(l(\tilde{\theta})\big)\,n_2(\tilde{\theta})\,d\tilde{\alpha} = |X_B(\tilde{\beta})|^2 \qquad (24)$$

with $n_1(\tilde{\theta}) = \big[r^A_{\tilde{\alpha}}\big(l_1(\tilde{\theta})\big)\big]^2\left|\frac{\partial}{\partial\tilde{\beta}}l_1(\tilde{\theta})\right|$, $n_2(\tilde{\theta}) = \big[R^B_{\tilde{\beta}}\big(l_2(\tilde{\theta})\big)\big]^2\left|\frac{\partial}{\partial\tilde{\alpha}}l_2(\tilde{\theta})\right|$. With (15)-(18), it can be shown that

$$n_1(\tilde{\theta}) = \big|J(\tilde{\theta})/\tilde{\mu}'_A(\tilde{\alpha})\big|\,, \qquad n_2(\tilde{\theta}) = \big|J(\tilde{\theta})/\tilde{\mu}'_B(\tilde{\beta})\big| \qquad (25)$$

where $J(\tilde{\theta}) = \frac{\partial l_1}{\partial\tilde{\alpha}}\frac{\partial l_2}{\partial\tilde{\beta}} - \frac{\partial l_2}{\partial\tilde{\alpha}}\frac{\partial l_1}{\partial\tilde{\beta}}$ is the Jacobian of $l(\tilde{\theta})$.

Characteristic Function Method. Following [10, 11], a class of QTFRs can be constructed as

$$\bar{T}_x(z) = \int_{\mathcal{D}} g(\theta)\,\langle x,\mathbf{D}_\theta\,x\rangle\,\Lambda\big(l^{-1}(z),\theta\big)\,d\theta \qquad (26)$$

with

$$\Lambda\big(\tilde{\theta},\theta\big) = \lambda^A_{\alpha,\tilde{\alpha}}\,\lambda^B_{\beta,\tilde{\beta}}\,|\mu'_A(\alpha)\,\mu'_B(\beta)|\,, \qquad (27)$$

where $g(\theta) = g(\alpha,\beta)$ is a kernel independent of $x(t)$ and $\langle x,\mathbf{D}_\theta\,x\rangle$ is the "characteristic function." If

$$g(\theta_\alpha) = g(\alpha,\beta_0) = 1 \quad\text{and}\quad g(\theta_\beta) = g(\alpha_0,\beta) = 1\,, \qquad (28)$$

then $\bar{T}_x(z)$ can be shown [10] to satisfy the marginal properties (generally different from (23), (24))

$$\int_{\tilde{\mathcal{B}}} \bar{T}_x\big(l(\tilde{\theta})\big)\,|\tilde{\mu}'_B(\tilde{\beta})|\,d\tilde{\beta} = |X_A(\tilde{\alpha})|^2 \qquad (29)$$

$$\int_{\tilde{\mathcal{A}}} \bar{T}_x\big(l(\tilde{\theta})\big)\,|\tilde{\mu}'_A(\tilde{\alpha})|\,d\tilde{\alpha} = |X_B(\tilde{\beta})|^2\,. \qquad (30)$$

5 THE CONJUGATE CASE

Two PDOs $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ with composition laws $\mathbf{A}_{\alpha_2}\mathbf{A}_{\alpha_1} = \mathbf{A}_{\alpha_1\bullet\alpha_2}$ and $\mathbf{B}_{\beta_2}\mathbf{B}_{\beta_1} = \mathbf{B}_{\beta_1\bullet\beta_2}$ are called *conjugate* [15] if[6]

$$\big(\mathbf{B}_\beta\,u^A_{\tilde{\alpha}}\big)(t) = u^A_{\tilde{\alpha}\bullet\beta}(t)\,, \qquad \big(\mathbf{A}_\alpha\,u^B_{\tilde{\beta}}\big)(t) = u^B_{\tilde{\beta}\bullet\alpha}(t)\,. \qquad (31)$$

This implies $(\mathcal{F}_A\mathbf{B}_\beta x)(\tilde{\alpha}) = (\mathcal{F}_A x)(\tilde{\alpha}\bullet\beta^{-1})$ and $(\mathcal{F}_B\mathbf{A}_\alpha x)(\tilde{\beta}) = (\mathcal{F}_B\,x)(\tilde{\beta}\bullet\alpha^{-1})$. Furthermore, using (14) we can show

Theorem 2. Conjugate PDOs $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$ commute up to a phase factor,

$$\mathbf{A}_\alpha\mathbf{B}_\beta = \lambda^A_{\alpha,\beta}\,\mathbf{B}_\beta\mathbf{A}_\alpha\,, \qquad (32)$$

and their eigenvalues and eigenfunctions are related as $\lambda^A_{\alpha,\beta} = \lambda^{B*}_{\beta,\alpha}$ and $\langle u^A_{\tilde{\alpha}}, u^B_{\tilde{\beta}}\rangle = \lambda^B_{\tilde{\alpha},\tilde{\beta}}$.

With (11), it follows that

$$\lambda^A_{\alpha,\tilde{\alpha}} = e^{\pm j2\pi\,\mu(\alpha)\,\mu(\tilde{\alpha})} \quad\text{and}\quad \lambda^B_{\beta,\tilde{\beta}} = e^{\mp j2\pi\,\mu(\beta)\,\mu(\tilde{\beta})}\,.$$

[6] Note that the groups and dual groups underlying $\mathbf{A}_\alpha$, $\mathbf{B}_\beta$ have to be identical: $(\mathcal{A},\bullet) = (\mathcal{B},*) = (\tilde{\mathcal{A}},\tilde{\bullet}) = (\tilde{\mathcal{B}},\tilde{*})$. Furthermore, the functions $\mu_A(\cdot), \mu_B(\cdot), \tilde{\mu}_A(\cdot)$, and $\tilde{\mu}_B(\cdot)$ are all equal up to sign factors, so that we will simply write $\mu(\cdot)$ in the following.

We now consider the composite operator $\mathbf{D}_\theta = \mathbf{D}_{\alpha,\beta} = \mathbf{B}_\beta \mathbf{A}_\alpha$. With (32), it is easily shown that $\mathbf{D}_\theta$ satisfies the central DO composition property (1),

$$\mathbf{D}_{\theta_2}\mathbf{D}_{\theta_1} = \lambda^A_{\alpha_2,\beta_1}\,\mathbf{D}_{\alpha_1\bullet\alpha_2,\,\beta_1\bullet\beta_2}\,, \tag{33}$$

as well as the relation

$$\mathbf{D}^{-1}_{\theta'}\,\mathbf{D}_\theta\,\mathbf{D}_{\theta'} \;=\; \lambda^A_{\alpha,\beta'}\,\lambda^B_{\beta,\alpha'}\,\mathbf{D}_\theta\,. \tag{34}$$

Eq. (33) implies that the separability condition (9) is met and that the group $(\mathcal{D},\circ)$ is *commutative*, $\theta_1\circ\theta_2 = \theta_2\circ\theta_1$.

We conjecture that, in the conjugate case, the DF and LF of $\mathbf{D}_\theta$ are related as $d\big(l(\tilde\alpha,\tilde\beta);\alpha,\beta\big) = l\big(\tilde\alpha\bullet\beta,\tilde\beta\bullet\alpha\big)$ or briefly

$$d\big(l(\tilde\theta),\theta\big) = l\big(\tilde\theta\circ\theta^T\big) \quad \text{with } \theta^T = (\alpha,\beta)^T \triangleq (\beta,\alpha)\,. \tag{35}$$

To motivate (35), recall that $z = l(\tilde\alpha,\tilde\beta)$ is the intersection of $\nu^A_{\tilde\alpha}(t)$ and $\tau^B_{\tilde\beta}(f)$. With (10) and (31), $\big(\mathbf{D}_\theta\, u^A_{\tilde\alpha}\big)(t) = \lambda^A_{\alpha,\tilde\alpha}\, u^A_{\tilde\alpha\bullet\beta}(t)$ and $\big(\mathbf{D}_\theta\, u^B_{\tilde\beta}\big)(t) = \lambda^B_{\beta,\tilde\beta\bullet\alpha}\, u^B_{\tilde\beta\bullet\alpha}(t)$. These signals are located along the curves $\nu^A_{\tilde\alpha\bullet\beta}(t)$ and $\tau^B_{\tilde\beta\bullet\alpha}(f)$, respectively, whose intersection is $z' = l(\tilde\alpha\bullet\beta,\tilde\beta\bullet\alpha)$. On the other hand, since z' has been derived from z through a displacement by θ, there should be $z' = d(z,\theta)$. This finally gives $d\big(l(\tilde\alpha,\tilde\beta);\alpha,\beta\big) = l(\tilde\alpha\bullet\beta,\tilde\beta\bullet\alpha)$. Note that the covariance (3) can now be rewritten as

$$T_{\mathbf{D}_\theta x}\big(l(\tilde\theta)\big) = T_x\big(l(\tilde\theta\circ\theta^{-T})\big) \quad \text{with } \theta^{-T} = (\beta^{-1},\alpha^{-1})\,.$$

Choosing, for simplicity, the reference TF point z_0 in (4)-(6) as $z_0 = l(\tilde\theta_0)$, (35) implies

$$l(\tilde\theta) = d\big(z_0,\tilde\theta^T\big) \quad \text{and} \quad p\big(l(\tilde\theta),z_0\big) = \tilde\theta^T\,. \tag{36}$$

Theorem 3. If $\mathbf{D}_\theta = \mathbf{B}_\beta\mathbf{A}_\alpha$ is a separable DO with conjugate PDOs $\mathbf{A}_\alpha$ and $\mathbf{B}_\beta$, and if (36) holds, then the $\mathbf{D}_\theta$-covariant QTFR class (6) equals the QTFR class (26). The kernels $h(t_1,t_2)$ in (6) and $g(\theta)$ in (26) are related as

$$h(t_1,t_2) = \int_{\mathcal{D}} g^*(\theta)\, D_\theta(t_1,t_2)\,|\mu'(\alpha)\,\mu'(\beta)|\,d\theta\,, \tag{37}$$

where $D_\theta(t_1,t_2)$ is the kernel of the DO $\mathbf{D}_\theta$.

Proof. The QTFR $\bar T_x(z)$ in (26) can be written as $\bar T_x(z) = \big\langle x,\,\bar{\mathbf{H}}^D_z x\big\rangle$ with $\bar{\mathbf{H}}^D_z = \int_{\mathcal{D}} g^*(\theta)\,\Lambda^*\big(l^{-1}(z),\theta\big)\,\mathbf{D}_\theta\,d\theta$. Comparing with (6), it remains to show that

$$\mathbf{D}_{p(z,z_0)}\mathbf{H}\,\mathbf{D}^{-1}_{p(z,z_0)} = \int_{\mathcal{D}} g^*(\theta)\,\Lambda^*\big(l^{-1}(z),\theta\big)\,\mathbf{D}_\theta\,d\theta$$

for all z. Setting $z = l(\tilde\theta)$, using (36), and multiplying by $\mathbf{D}^{-1}_{\tilde\theta^T}$ and $\mathbf{D}_{\tilde\theta^T}$ from left and right, respectively, this becomes

$$\begin{aligned}\mathbf{H} &= \int_{\mathcal{D}} g^*(\theta)\,\Lambda^*\big(\tilde\theta,\theta\big)\,\mathbf{D}^{-1}_{\tilde\theta^T}\,\mathbf{D}_\theta\,\mathbf{D}_{\tilde\theta^T}\,d\theta \\ &= \int_{\mathcal{D}} g^*(\theta)\,|\lambda^A_{\alpha,\tilde\alpha}|^2|\lambda^B_{\beta,\tilde\beta}|^2\,|\mu'(\alpha)\,\mu'(\beta)|\,\mathbf{D}_\theta\,d\theta\end{aligned}$$

where (27) and (34) have been used. With $|\lambda^A_{\alpha,\tilde\alpha}|^2 = |\lambda^B_{\beta,\tilde\beta}|^2 = 1$, we obtain $\mathbf{H} = \int_{\mathcal{D}} g^*(\theta)\,|\mu'(\alpha)\,\mu'(\beta)|\;\mathbf{D}_\theta\,d\theta$, which is (37), and relates the kernels $h(t_1,t_2)$ and $g(\alpha,\beta)$ independently of the external parameter $\tilde\theta$. ∎

Theorem 3 states that the covariance approach and the characteristic function method are equivalent in the conjugate case. Two important conclusions can now be drawn:

- The $\mathbf{D}_\theta$-covariant QTFR class in (4)-(6) satisfies the marginal properties[7] (29), (30) if the simple kernel constraint (28) is met.
- The QTFR class (26) obtained with the characteristic function method satisfies the $\mathbf{D}_\theta$-covariance (3).

Examples. The PDOs $\mathbf{T}_\tau$ and $\mathbf{F}_\nu$ underlying *Cohen's class* (7) are conjugate. Hence, Cohen's class can be constructed using either the covariance method or the characteristic function method. It is $\mathbf{S}_{\tau,\nu}$-covariant and (assuming that (28) is met) it satisfies also the marginal properties. An analogous result holds for the *hyperbolic class* [6].

The PDOs $\mathbf{L}_a$ and $\mathbf{T}_\tau$ underlying the *affine class* (8) are *not* conjugate. Hence, the characteristic function method yields a class [11] that is different from the affine class and that is not $\mathbf{C}_{a,\tau}$-covariant. Similarly, the *power classes* [7, 8] are also based on non-conjugate operators.

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[7] Due to (25), the marginal properties (29), (30) will be identical to the marginal properties (23), (24) and, in turn, (19), (20) if and only if the LF's Jacobian is $J(\tilde\theta) = \pm\tilde\mu'_A(\tilde\alpha)\,\tilde\mu'_B(\tilde\beta)$. We conjecture that, in the conjugate case, this relation is always satisfied and the two sets of marginal properties are thus equivalent.