A UNIFIED FRAMEWORK FOR THE SCALE COVARIANT AFFINE, HYPERBOLIC, AND POWER CLASS QUADRATIC TIME-FREQUENCY REPRESENTATIONS USING GENERALIZED TIME SHIFTS*

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ABSTRACT

We propose a framework that unifies and extends the affine, hyperbolic, and power classes of quadratic time-frequency representations (QTFRs). These QTFR classes satisfy the scale covariance property, important in multiresolution analysis, and a generalized time-shift covariance property, important in the analysis of signals propagating through dispersive systems. We provide a general class formulation in terms of 2-D kernels, a generalized signal expansion, a list of desirable QTFR properties with kernel constraints, and a "central QTFR" generalizing the Wigner distribution and the Altes-Marinovich Q-distribution. We also propose two generalized time-shift covariant (not, in general, scale covariant) QTFR classes by applying a generalized warping to Cohen's class and to the affine class.

1. INTRODUCTION

Quadratic time-frequency representations (QTFRs) have been used successfully for the analysis of nonstationary signals [1]. Different QTFRs are best suited for analyzing different types of signals. These QTFRs can be classified based on the set of properties they satisfy. Two major QTFR classes are Cohen's class of time-frequency (TF) shift covariant QTFRs [2, 1], and the affine class of time-shift and scale covariant QTFRs [3, 4, 1]. Recently, we have proposed the hyperbolic class [5], and the power classes [6] which include the affine class. These QTFRs satisfy scale covariance and a specific dispersive time-shift covariance. In this paper, we propose a generalized QTFR class that unifies and extends the existing scale covariant QTFR classes [7, 8].

2. GENERALIZED QTFR CLASS

The generalized class consists of all QTFRs, $T_X^{(G)}(t,f)$, that satisfy the scale covariance property and the generalized time-shift covariance property defined, respectively, as

$$T^{(G)}_{\mathcal{C}_a X}(t,f) = T^{(G)}_X(at, f/a) \quad (1)$$

$$T^{(G)}_{\mathcal{D}_c X}(t,f) = T^{(G)}_X\big(t - c\,\tau(f), f\big), \quad (2)$$

where $X(f)$ is the Fourier transform of a signal $x(t)$, the scale operator $\mathcal{C}_a$ and the generalized time-shift operator $\mathcal{D}_c$ are defined as $(\mathcal{C}_a X)(f) = X(\frac{f}{a})/\sqrt{|a|}$ and $(\mathcal{D}_c X)(f) = e^{-j2\pi c \xi(f/f_r)} X(f)$ where $f_r > 0$ is a fixed reference frequency, $\xi(b)$ is a given phase function that is assumed one-to-one, and the group delay function, up to a factor c, is

$$\tau(f) = \frac{1}{f_r}\,\xi'\left(\frac{f}{f_r}\right) \quad \text{with} \quad \xi'(b) = \frac{d}{db}\xi(b).$$

Scale covariance is important in multiresolution analysis, and generalized time-shift covariance is useful for analyzing signals propagating through systems with dispersion characteristics corresponding to group delays of the form $c\tau(f)$. Note that a specific QTFR class will be obtained for a specific choice of the basic group delay function $\tau(f)$ in (2).

It can be shown [6, 7] that a QTFR $T_X^{(G)}(t,f)$ satisfying the covariances (1)-(2) exists for a given phase function $\xi(b)$ (or, equivalently, group delay $\tau(f)$) if and only if a "kernel" $\Gamma_{T^{(G)}}(b_1,b_2)$ can be found that satisfies the condition

$$\Gamma_{T^{(G)}}(b_1,b_2)\,\frac{\xi(\alpha b_1)-\xi(\alpha b_2)}{\alpha\xi'(\alpha)} \equiv \Gamma_{T^{(G)}}(b_1,b_2)\,\frac{\xi(b_1)-\xi(b_2)}{\xi'(1)} \quad (3)$$

for all b_1, b_2, α; this QTFR can then be written in terms of the kernel $\Gamma_{T^{(G)}}(b_1,b_2)$ as

$$T_X^{(G)}(t,f) = \frac{1}{|f|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma_{T^{(G)}}\left(\frac{f_1}{f},\frac{f_2}{f}\right) X(f_1)\,X^*(f_2) \cdot e^{j2\pi\frac{tf}{f_r\tau(f_r)}\left[\xi\left(\frac{f_1}{f}\right)-\xi\left(\frac{f_2}{f}\right)\right]}\,df_1 df_2. \quad (4)$$

It can further be shown [9, 7] that condition (3) is satisfied for *arbitrary* kernel $\Gamma_{T^{(G)}}(b_1,b_2)$ if and only if the group delay is a *power function*, which leads to the hyperbolic class and the power classes [5, 6]. However, (3) is also satisfied for the phase function $\xi(b) = b\ln b$ (corresponding to the group delay $\tau(f) = [1+\ln(f/f_r)]/f_r$) and kernel $\Gamma_{P_1}(b_1,b_2) = \int_{-\infty}^{\infty}\delta\big(b_1-\lambda_1(u)\big)\,\delta\big(b_2-\lambda_1(-u)\big)\,\mu(u)\,du$ with $\lambda_1(u) = \exp\left(1+\frac{u\,e^{-u}}{e^{-u}-1}\right)$ which defines the Bertrand P_1-distributions [4]. Note that for this phase function, $\Gamma_{P_1}(b_1,b_2)$ cannot take on arbitrary form, but it is parameterized by a real and even 1-D weighting function $\mu(u)$. Some important aspects of the generalized class follow.

Generalized signal expansion. The TF geometry underlying the generalized QTFR class (4) is related to the *generalized impulse*

$$I_c(f) = \sqrt{|\tau(f)|}\,e^{-j2\pi c\,\xi\left(\frac{f}{f_r}\right)}, \quad c \in \mathbb{R} \quad (5)$$

with spectral energy density $|I_c(f)|^2 = |\tau(f)|$ and group delay $c\tau(f)$. Note that generalized time-shifting a general-

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ized impulse $I_c(f)$ simply changes its parameter value, i.e. $(\mathcal{D}_{c_0} I_c)(f) = I_{c+c_0}(f)$. If $\xi(b)$ is one-to-one with domain $\mathbb{R}$ and range $\mathbb{R}$, then any finite energy signal $X(f)$ can be expanded in terms of generalized impulses,

$$X(f) = \int_{-\infty}^{\infty} \rho_X(c)\, I_c(f)\, dc\,, \tag{6}$$

where the coefficient function $\rho_X(c)$ is the inner product

$$\rho_X(c) = \int_{-\infty}^{\infty} X(f)\, I_c^*(f)\, df\,.$$

The coefficient function of a generalized impulse $I_{c_0}(f)$ is a Dirac delta function at $c{=}c_0$, i.e. $\rho_{I_{c_0}}(c) = \delta(c - c_0)$.

Properties and kernel constraints. A list of desirable properties which one might want generalized QTFRs to satisfy, with corresponding kernel constraints, is given below:

P-1 Scale covariance:
$T^{(G)}_{\mathcal{C}_a X}(t,f) = T^{(G)}_X(at, f/a)$: always satisfied

P-2 Generalized time-shift covariance:
$T^{(G)}_{\mathcal{D}_c X}(t,f) = T^{(G)}_X\big(t - c\,\tau(f), f\big)$: always satisfied

P-3 Real-valuedness:
$T^{(G)*}_X(t,f) = T^{(G)}_X(t,f)$ if $\Gamma_{T^{(G)}}(b_1,b_2) = \Gamma^*_{T^{(G)}}(b_2,b_1)$

P-4 Energy distribution: $\iint T^{(G)}_X(t,f)\, dt df = \int |X(f)|^2 df$
if $\int \Gamma_{T^{(G)}}(b,b) \left|\frac{\tau(f/b)}{b\tau(f)}\right| db = 1\ \forall f$

P-5 Frequency marginal:
$\int T^{(G)}_X(t,f)\, dt = |X(f)|^2$ if $\Gamma_{T^{(G)}}(b,b) = \delta(b-1)$

P-6 Generalized marginal: $\int T^{(G)}_X(c\tau(f), f)|\tau(f)| df = |\rho_X(c)|^2$
if $\int \Gamma_{T^{(G)}}\big(b, \frac{f_2}{f_1} b\big) \frac{|\tau(f_1/b)|}{\sqrt{|\tau(f_1)\tau(f_2)|}} \frac{db}{|b|} = 1,\ \forall f_1, f_2$

P-7 Frequency localization:
$X(f) = \delta(f - \hat{f}) \Rightarrow T^{(G)}_X(t,f) = \delta(f - \hat{f})$ if $\Gamma_{T^{(G)}}(b,b) = \delta(b-1)$

P-8 Generalized localization: $T^{(G)}_{I_c}(t,f) = |\tau(f)|\delta(t - c\tau(f))$
if $\int \Gamma_{T^{(G)}}\big(\frac{d(b,\beta)}{\alpha}, \frac{d(b,-\beta)}{\alpha}\big) \frac{db}{\sqrt{|\xi'(d(b,\beta))\xi'(d(b,-\beta))|}} = |\alpha|, \forall \alpha, \beta$
where $d(b,\beta) = \xi^{-1}(b + \frac{\beta}{2})$, and the inverse phase function, $\xi^{-1}(b)$, is such that $\xi(\xi^{-1}(b)) = b$.

Additional properties and constraints can be found in [7, 8].

Central QTFR. A "central QTFR" of the generalized class can be defined as

$$Q^{(G)}_X(t,f) = |f| \int_{-\infty}^{\infty} X\big(f\,g(\beta)\big)\, X^*\big(f\,g(-\beta)\big)\, e^{j2\pi \frac{tf}{f_r \tau(f_r)}\beta} \cdot \left| \frac{\xi'\big(\sqrt{|g(\beta)\,g(-\beta)|}\big)}{\xi'(g(\beta))\,\xi'(g(-\beta))} \right| d\beta \tag{7}$$

with $g(\beta) = \xi^{-1}\big(\xi(1) + \frac{\beta}{2}\big)$. The 2-D kernel of $Q^{(G)}_X(t,f)$ is $\Gamma_{Q^{(G)}}(b_1,b_2) = \big|\xi'\big(\sqrt{|b_1 b_2|}\big)\big|\, \delta\big(\frac{\xi(b_1)+\xi(b_2)}{2} - \xi(1)\big)$. The central QTFR satisfies a large number of desirable properties such as properties P-1–P-5, and P-7; it will also satisfy P-6 and P-8 when the phase function $\xi(b)$ is a logarithmic or power function. The central QTFR simplifies to the Wigner distribution (WD) [10] of the affine class when $\xi(b) = b$ and to the Altes-Marinovich Q-distribution of the hyperbolic class [5] when $\xi(b) = \ln b$.

3. GENERALIZED CLASS EXAMPLES

The generalized class in (4) depends on the specific group delay $\tau(f)$ or, equivalently, phase function $\xi(b)$, whose choice is constrained by condition (3). Some important examples are discussed below which correspond to all phase functions that satisfy (3) with *arbitrary* kernel $\Gamma_{T^{(G)}}(b_1,b_2)$. These examples are summarized in Table 1.

3.1. Power QTFR Classes

The family of *power classes*, with real-valued power $\kappa \neq 0$, are obtained with $\xi(b) = \xi_\kappa(b) = \mathrm{sgn}(b)\,|b|^\kappa$ (where $\mathrm{sgn}(b)$ is $+1$ if $b > 0$, -1 if $b < 0$) and $\tau(f) = \tau_\kappa(f) = (\kappa/f_r)|f/f_r|^{\kappa-1}$ [6]. The power class QTFRs, $T^{(\kappa)}_X(t,f)$, are scale and power time-shift covariant as in (1)-(2) with the generalized time-shift operator, $\mathcal{D}_c$, simplified to the power time-shift operator $\mathcal{D}^{(\kappa)}_c$ defined as $(\mathcal{D}^{(\kappa)}_c X)(f) = e^{-j2\pi c\, \xi_\kappa(f/f_r)}\, X(f)$. Any power class QTFR can be written in terms of an *arbitrary* 2-D kernel $\Gamma^{(\kappa)}_T(b_1,b_2)$ as in (4),

$$T^{(\kappa)}_X(t,f) = \frac{1}{|f|} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma^{(\kappa)}_T\Big(\frac{f_1}{f}, \frac{f_2}{f}\Big)\, X(f_1)\, X^*(f_2) \cdot e^{j2\pi \frac{tf}{\kappa}\left[\xi_\kappa\left(\frac{f_1}{f}\right) - \xi_\kappa\left(\frac{f_2}{f}\right)\right]} df_1 df_2\,.$$

The power impulse used in the signal expansion in (6) is $I^{(\kappa)}_c(f) = \sqrt{|\tau_\kappa(f)|}\, e^{-j2\pi c\, \xi_\kappa(\frac{f}{f_r})}$ (cf. (5)). The central QTFR (7) of the power classes is the *power WD*, $Q^{(\kappa)}_X(t,f)$, with kernel $\Gamma^{(\kappa)}_{Q^{(\kappa)}}(b_1,b_2) = \big|\xi'_\kappa\big(\sqrt{|b_1 b_2|}\big)\big|\, \delta\big(\frac{\xi_\kappa(b_1)+\xi_\kappa(b_2)}{2} - 1\big)$ [6]. The power WD satisfies properties P-1–P-8. Other important power classes QTFRs are the *Bertrand P_κ-distributions* [4, 6] and the *powergram* [6, 7].

The power class QTFRs can also be obtained by applying a unitary warping [6, 7, 11, 12] to the affine class QTFRs, $T^{(A)}(t,f)$, (see Subsection 3.2)

$$T^{(\kappa)}_X(t,f) = T^{(A)}_{\mathcal{W}_\kappa X}\Big(\frac{t}{f_r \tau_\kappa(f)},\, f_r\, \xi_\kappa\Big(\frac{f}{f_r}\Big)\Big) \tag{8}$$

where $(\mathcal{W}_\kappa X)(f) = X\big(f_r\, \xi_\kappa^{-1}(\frac{f}{f_r})\big) \Big/ \sqrt{f_r\, \big|\tau_\kappa\big(f_r\, \xi_\kappa^{-1}(\frac{f}{f_r})\big)\big|}$ is the power warped signal. For example, the power WD and the powergram are the warped versions of the WD and the scalogram of the affine class, respectively [6]. The warping converts the scale and non-dispersive time-shift covariances of the affine class to the scale and power time-shift covariances of the power classes, respectively, since $\mathcal{W}_\kappa^{-1} \mathcal{S}_{c/f_r} \mathcal{W}_\kappa = \mathcal{D}^{(\kappa)}_c$ and $\mathcal{W}_\kappa^{-1} \mathcal{C}_{\xi_\kappa(a)} \mathcal{W}_\kappa = \mathcal{C}_a$, where the time-shift operator is defined as $(\mathcal{S}_{\hat{t}} X)(f) = e^{-j2\pi \hat{t} f} X(f)$, and $\mathcal{W}_\kappa^{-1}$ is such that $(\mathcal{W}_\kappa^{-1} \mathcal{W}_\kappa X)(f) = X(f)$.

3.2. Affine QTFR Class

The *affine class* [3, 4] is the power class when $\kappa{=}1$; it is obtained from the general framework with $\xi(b) = \xi_1(b) = b$ and constant group delay $\tau(f) = \tau_1(f) = 1/f_r$. The affine QTFRs, $T^{(A)}_X(t,f)$, are scale and constant time-shift covariant as in (1)-(2) with the generalized time-shift operator $\mathcal{D}_c$ reduced to the conventional (non-dispersive) time-shift operator $\mathcal{S}_{\hat{t}}$. Any QTFR of the affine class can be written in terms of an *arbitrary* 2-D kernel $\Gamma^{(A)}_T(b_1,b_2)$ [3] as (cf. (4))

$$T^{(A)}_X(t,f) = \frac{1}{|f|} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma^{(A)}_T\Big(\frac{f_1}{f}, \frac{f_2}{f}\Big)\, X(f_1)\, X^*(f_2)$$

QTFR class:	Generalized class $T_X^{(G)}(t,f)$	Power classes $T_X^{(\kappa)}(t,f)$	Affine class $T_X^{(A)}(t,f)$	Chirp class $T_X^{(2)}(t,f)$	Hyperbolic class $T_X^{(H)}(t,f)$
phase function, $\xi(b)$:	$\xi(b)$ as in (3)	$\xi_\kappa(b)=\mathrm{sgn}(b)\,\lvert b\rvert^\kappa$	b	$\mathrm{sgn}(b)\,\lvert b\rvert^2$	$\ln b$
group delay, $\tau(f)$:	$\tau(f)$	$\tau_\kappa(f)=\frac{\kappa}{f_r}\left\lvert\frac{f}{f_r}\right\rvert^{\kappa-1}$	$1/f_r$	$2\frac{\lvert f\rvert}{f_r^2}$	$1/f$
covariances with respect to:	• scaling, $\mathcal{C}_a$ • generalized time-shift, $\mathcal{D}_c$	• scaling, $\mathcal{C}_a$ • power time-shift, $\mathcal{D}_c^{(\kappa)}$	• scaling, $\mathcal{C}_a$ • constant time-shift, $\mathcal{S}_{c/f_r}$	• scaling, $\mathcal{C}_a$ • chirp time-shift, $\mathcal{D}_c^{(2)}$	• scaling, $\mathcal{C}_a$ • hyperbolic time-shift, $\mathcal{H}_c$
central QTFR:	generalized WD $Q_X^{(G)}(t,f)$	power WD $Q_X^{(\kappa)}(t,f)$	WD $W_X(t,f)$	chirp WD $Q_X^{(2)}(t,f)$	Q-distribution $Q_X^{(H)}(t,f)$

Table 1: *Generalized time-shift covariant and scale covariant QTFR classes, their corresponding phase and group delay functions, pair of covariance properties, and central QTFR.*

$$\cdot\, e^{j2\pi t(f_1-f_2)}\, df_1 df_2.$$

The central QTFR (7) of the affine class is the *WD* [10, 1]

$$W_X(t,f) = \int_{-\infty}^{\infty} X\left(f+\frac{\nu}{2}\right) X^*\left(f-\frac{\nu}{2}\right) e^{j2\pi t\nu}\, d\nu$$

with kernel $\Gamma_W^{(A)}(b_1,b_2)=\delta\left(\frac{b_1+b_2}{2}-1\right)$.

3.3. Chirp QTFR Class

The *chirp class* is the power class when $\kappa=2$, and the generalized class with quadratic phase $\xi(b)=\xi_2(b)=\mathrm{sgn}(b)\,|b|^2$ and linear group delay $\tau(f)=\tau_2(f)=2|f|/f_r^2$. The chirp class QTFRs, $T_X^{(2)}(t,f)$, are scale covariant and chirp time-shift covariant. The chirp time-shift operator $\mathcal{D}_c^{(2)}$ produces time delays that are proportional to $|f|$. The central QTFR (7) of the chirp class is the *chirp WD*, $Q_X^{(2)}(t,f)$, obtained with kernel $\Gamma_{Q^{(2)}}^{(2)}(b_1,b_2)=2\sqrt{|b_1b_2|}\,\delta\left(\frac{\xi_2(b_1)+\xi_2(b_2)}{2}-1\right)$.

3.4. Hyperbolic QTFR Class

The *hyperbolic class* is the generalized class with $\xi(b)=\ln b$ and hyperbolic group delay $\tau(f)=1/f$ [5]. It is defined for analytic signals (i.e., $X(f)=0$ for $f<0$). The hyperbolic class QTFRs, $T_X^{(H)}(t,f)$, are scale and time-shift covariant as in (1)-(2) with the generalized time-shift operator $\mathcal{D}_c$ simplified to the hyperbolic time-shift operator $\mathcal{H}_c$ that is defined as $(\mathcal{D}_cX)(f)=(\mathcal{H}_cX)(f)=e^{-j2\pi c\ln(f/f_r)}X(f)$. The hyperbolic class QTFRs can be written in terms of an *arbitrary* 2-D kernel $\Gamma_T^{(H)}(b_1,b_2)$ as (cf. (4))

$$T_X^{(H)}(t,f) = \frac{1}{f}\int_0^{\infty}\int_0^{\infty} \Gamma_T^{(H)}\left(\frac{f_1}{f},\frac{f_2}{f}\right) X(f_1)X^*(f_2) \cdot e^{j2\pi t f\ln\frac{f_1}{f_2}}\, df_1 df_2, \quad f>0.$$

The hyperbolic impulse, used in the signal expansion in (6), is $I_c^{(H)}(f)=(1/\sqrt{f})\,e^{-j2\pi c\ln(f/f_r)}, f>0$ (cf. (5)). The central QTFR (7) of the hyperbolic class is the *Altes-Marinovich Q-distribution* [5]

$$Q_X^{(H)}(t,f) = f\int_{-\infty}^{\infty} X\left(f\,e^{\beta/2}\right) X^*\left(f\,e^{-\beta/2}\right) e^{j2\pi tf\beta}\, d\beta$$

with kernel $\Gamma_{Q^{(H)}}^{(H)}(b_1,b_2)=2\delta\left(\ln(b_1b_2)\right)/\sqrt{b_1b_2}$. The Q-distribution satisfies properties P-1–P-8.

The hyperbolic class QTFRs are also obtained by warping Cohen's class QTFRs, $T_X^{(C)}(t,f)$, according to [5]

$$T_X^{(H)}(t,f) = T_{\mathcal{W}_H X}^{(C)}\left(\frac{tf}{f_r}, f_r\ln\frac{f}{f_r}\right), \quad f>0 \qquad (9)$$

with the warped signal $(\mathcal{W}_HX)(f)=\sqrt{e^{f/f_r}}\,X\left(f_re^{f/f_r}\right), \forall f$. This warping changes the constant-bandwidth nature of Cohen's class QTFRs [1] to the constant-Q nature of the hyperbolic class QTFRs [5]. For example, the WD and spectrogram of Cohen's class map to the Q-distribution and hyperbologram, respectively, of the hyperbolic class [5]. The warping maps the time and frequency shift covariances of Cohen's class to the scale and hyperbolic time-shift covariances of the hyperbolic class, respectively, since $\mathcal{W}_H^{-1}\mathcal{S}_{c/f_r}\mathcal{W}_H=\mathcal{H}_c$ and $\mathcal{W}_H^{-1}\mathcal{M}_{f_r\ln a}\mathcal{W}_H=\mathcal{C}_a$ where the frequency-shift operator is defined as $(\mathcal{M}_\nu X)(f)=X(f-\nu)$.

4. GENERALIZED TIME-SHIFT COVARIANT QTFR CLASSES

The generalized QTFR class discussed so far is axiomatically defined by the scale covariance property (1) and the generalized time-shift covariance property (2). However, the admissible phase functions $\xi(b)$ are restricted since condition (3) must be satisfied. If scale covariance is not of interest, and only generalized time-shift covariance is important, then other QTFR generalizations are possible [7, 8]. We can obtain generalized time-shift covariant QTFRs by warping Cohen's class or the affine class[1] using

$$T_X^{(W\,class)}(t,f) = T_{\mathcal{W}X}^{(class)}\left(\frac{t}{f_r\tau(f)}, f_r\,\xi\left(\frac{f}{f_r}\right)\right), \qquad (10)$$

where the generalized warping operator $\mathcal{W}$ is defined by $(\mathcal{W}X)(f)=X\left(f_r\xi^{-1}\left(\frac{f}{f_r}\right)\right)\Big/\sqrt{f_r\left|\tau\left(f_r\xi^{-1}\left(\frac{f}{f_r}\right)\right)\right|}$. The QTFR class, $(class)$, which undergoes the warping is either Cohen's class, (C), or the affine class, (A). It is transformed either into the QTFRs $T_X^{(WC)}(t,f)$ (warped Cohen's class) or $T_X^{(WA)}(t,f)$ (warped affine class), respectively, both of which satisfy the generalized time-shift covariance (2). In (10), the phase function $\xi(b)$ need not satisfy condition (3). The generalized time-shift covariant QTFR classes thus obtained are discussed below and summarized in Table 2 [7, 8].

Warped Cohen's class. By warping the QTFRs of Cohen's class, $T_X^{(C)}(t,f)$, using (10) with $(class)=(C)$, we obtain the generalized time-shift covariant QTFRs

$$T_X^{(WC)}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_X^{(W)}(\hat{t},\hat{f})$$

[1]Unitary warpings from Cohen's class or the affine class have been proposed in [11, 12]. Here, we consider the special case of warpings that lead to generalized time-shifts.

Warped Cohen's class, $T_X^{(WC)}(t,f)$	Warped affine class, $T_X^{(WA)}(t,f)$
covariances w.r.t.: • warped frequency shift, $\mathcal{W}^{-1}\mathcal{M}_\nu\mathcal{W}$ • generalized time shift, $\mathcal{D}_c$	covariances w.r.t.: • warped scaling, $\mathcal{W}^{-1}\mathcal{C}_a\mathcal{W}$ • generalized time shift, $\mathcal{D}_c$
Prominent example: Hyperbolic class $\xi(b)=\ln b$, $\tau(f)=1/f$	Prominent example: Power classes $\xi(b)=\xi_\kappa(b)$, $\tau(f)=\tau_\kappa(f)$

Table 2: *Generalized time-shift covariant classes, obtained by warping Cohen's class or the affine class.*

$$\cdot\, \psi_T^{(WC)}\left(\frac{t}{\tau(f)}-\frac{\hat{t}}{\tau(\hat{f})},\, \xi\left(\frac{f}{f_r}\right)-\xi\left(\frac{\hat{f}}{f_r}\right)\right) d\hat{t}\, d\hat{f}. \quad (11)$$

Here, $\psi_T^{(WC)}(c,b)$ is a 2-D kernel characterizing $T^{(WC)}$, and $Q_X^{(W)}(t,f)$, the generalized time-shift version of the WD (obtained by warping the WD using (10)), is given by

$$Q_X^{(W)}(t,f) = \int_{-\infty}^{\infty} X\left(f_r\, \hat{d}(f,\beta)\right) X^*\left(f_r\, \hat{d}(f,-\beta)\right) \cdot e^{j2\pi \frac{t}{\tau(f)}\beta} \frac{f_r}{\sqrt{\left|\xi'\left(\hat{d}(f,\beta)\right)\xi'\left(\hat{d}(f,-\beta)\right)\right|}}\, d\beta \quad (12)$$

where $\hat{d}(f,\beta)=\xi^{-1}\left(\xi(\frac{f}{f_r})+\frac{\beta}{2}\right)$. The generalized warped WD, $Q_X^{(W)}(t,f)$, in (12) is a member of the warped class (11) with kernel $\psi_Q^{(WC)}(c,b)=\delta(c)\delta(b)$. It is generally different from the generalized WD $Q_X^{(G)}(t,f)$ in (7) but coincides with $Q_X^{(G)}(t,f)$ for $\xi(b)=\xi_\kappa(b)$ or $\xi(b)=\ln b$. Hence, *both* $Q_X^{(W)}(t,f)$ and $Q_X^{(G)}(t,f)$ simplify to the WD when $\tau(f)=1/f_r$, to the Q-distribution when $\tau(f)=1/f$, and to the power WD when $\tau(f)=\tau_\kappa(f)$ [7]. The generalized warped WD, $Q_X^{(W)}(t,f)$, satisfies properties P-2–P-8.

The warping in (10) maps the non-dispersive (i.e. constant) time-shift covariance of Cohen's class to the generalized time-shift covariance since $\mathcal{W}^{-1}\mathcal{S}_{c/f_r}\mathcal{W}=\mathcal{D}_c$ [7, 8, 11]. However, it does not map the frequency-shift covariance of Cohen's class to scale covariance, except when the group delay is hyperbolic, $\tau(f)=1/f$. Hence, the warped Cohen's class in (10) coincides with the hyperbolic class in (9) when $(class)=(C)$, $\xi(b)=\ln b$, $\tau(f)=1/f$, and $\mathcal{W}=\mathcal{W}_H$.

Warped affine class. By warping the QTFRs of the affine class, $T_X^{(A)}(t,f)$, using (10) with $(class)=(A)$, we obtain the generalized time-shift covariant QTFRs

$$T_X^{(WA)}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_X^{(W)}(\hat{t},\hat{f}) \cdot \psi_T^{(WA)}\left(\xi\left(\frac{f}{f_r}\right)\left(\frac{t}{\tau(f)}-\frac{\hat{t}}{\tau(\hat{f})}\right),\, \frac{\xi(\frac{\hat{f}}{f_r})}{\xi(\frac{f}{f_r})}\right) d\hat{t}\, d\hat{f}. \quad (13)$$

Here, $\psi_T^{(WA)}(c,b)$ is a 2-D kernel characterizing $T^{(WA)}$, and $Q_X^{(W)}(t,f)$ is the warped WD in (12) with kernel $\psi_Q^{(WA)}(c,b)=\delta(c)\delta(b-1)$. Note that we obtain the same $Q_X^{(W)}(t,f)$ as in the warped Cohen's class since the WD is a member of both the affine class and Cohen's class. Hence, $Q_X^{(W)}(t,f)$ is a member of both classes (11) and (13).

The warping in (10) maps the non-dispersive time-shift covariance of the affine class to the generalized time-shift covariance. However, it does not preserve the scale covariance property of the affine class, except for power group delays $\tau(f)=\tau_\kappa(f)=(\kappa/f_r)\,|f/f_r|^{\kappa-1}$. Hence, the warped affine class in (10) coincides with a power class in (8) when $(class)=(A)$, $\xi(b)=\xi_\kappa(b)$, $\tau(f)=\tau_\kappa(f)$, and $\mathcal{W}=\mathcal{W}_\kappa$.

5. CONCLUSION

We proposed a generalized QTFR class that consists of all scale covariant and generalized time-shift covariant QTFRs. These QTFRs depend on a given group delay $\tau(f)$ or, equivalently, a given phase function $\xi(b)$. The phase function is constrained to satisfy condition (3). The generalized class framework unifies the power classes (including the affine and chirp classes) and the hyperbolic class (see Table 1).

We also proposed QTFR classes that are generalized time-shift covariant but, in general, not scale covariant (see Table 2). These QTFRs depend on an arbitrary group delay and a corresponding phase function, and are obtained by warping Cohen's class or the affine class. Using the generalized warping with a hyperbolic group delay, Cohen's class is mapped to the hyperbolic class, and with a power group delay, the affine class is mapped to the power classes. Thus, the hyperbolic and power classes are the special cases where both generalized class frameworks coincide.

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