# INSTANTANEOUS FREQUENCY AND TIME-FREQUENCY DISTRIBUTIONS

*Berkant Tacer and Patrick J. Loughlin*

Dept. of Electrical Engineerıng
University of Pittsburgh
Pittsburgh. PA 15261

## ABSTRACT

It is generally stated that the conditional mean frequency of a time-frequency distribution (TFD) should equal the instantaneous frequency of the signal. The commonly accepted definition of instantaneous frequency as the derivative of the phase of the analytic signal sometimes leads to curious results. Although it is commonly held that positivity of the TFD and satisfaction of the so-called "instantaneous frequency constraint" are generally incompatible, we show that one can *always* find a complex signal, the real part of which is the given signal, for which the derivative of the phase is consistent with the marginals and positivity. Furthermore, for the cases considered, the derivative of the phase of this signal, which by design equals the conditional mean frequency of a positive TFD, is a reasonable, readily interpretable choice for instantaneous frequency.

## INTRODUCTION

Instantaneous frequency is a fundamental concept not only in communication engineering (i.e., frequency modulation), but also in nature (e.g., light of changing color). Many definitions have been given (e.g., see [11], [14]), but it is commonly accepted that instantaneous frequency is the derivative of the phase of the signal.

One of the earliest methods for obtaining the phase of a signal was the quadrature method, which forms the complex signal $z(t) = A(t)e^{j\varphi(t)}$ from the real signal $s(t) = A(t)\cos(\varphi(t))$. From the complex signal, the phase is readily obtained, and the instantaneous frequency is defined as $\dot{\varphi}(t) = \frac{d}{dt}\varphi(t)$. The quadrature method, however, begs the issue, because it requires that we first write $s(t)$ as $A(t)\cos(\varphi(t))$, and there are an infinite number of possibilities for $A(t)$ and $\varphi(t)$. Gabor proposed a method for unambiguously defining the amplitude and phase by generating a specific complex signal, namely the analytic signal, from the given, real signal [8].[1]

Sometimes the instantaneous frequency, defined as the derivative of the phase, presents paradoxes that are difficult to reconcile with the physical nature of the signal and spectrum. For example, the derivative of the phase of the analytic signal can be negative, yet there are no negative frequencies in an analytic signal. More generally, the derivative of the phase of a complex signal (not necessarily analytic) can be outside the bandwidth of the signal. It is perplexing and difficult to understand how the "instantaneous frequency" of a band-limited signal can be outside the band. See [4] for discussion of some other paradoxes of instantaneous frequency.

In the time-frequency literature, instantaneous frequency is interpreted as the average frequency at each time [3]. The average frequency at each time is calculated from the time-conditional frequency distribution $P(\omega|t)$ of the signal as[2]

$$\langle\omega\rangle_t = \int \omega P(\omega|t)\,d\omega = \frac{1}{p(t)}\int \omega P(t,\omega)\,d\omega, \qquad (1)$$

where $p(t) = \int P(t,\omega)\,d\omega$ is the time marginal of the joint time-frequency distribution $P(t,\omega)$. It is generally stated that this conditional moment should equal the derivative of the phase of the complex (typically analytic) signal,

$$\langle\omega\rangle_t = \dot{\varphi}(t). \qquad (2)$$

The constraint on the distribution necessary for eq. (2) to hold is well known [2b], and an infinite number of distributions yield this result for *any* complex signal, not just the analytic signal [3]. This result suggests that time-frequency distributions offer a general approach to defining instantaneous frequency [2a]. At the very least, it certainly gives pause to insisting *a priori* on a particular form for the complex signal. If the derivative of the phase of a particular complex signal is difficult to interpret as "the average frequency at each time," for example, when it extends outside the bandwidth of the signal, why insist that the TFD yield this result for the conditional mean frequency?

It has been stated that positivity of the distribution is generally incompatible with eq. (2) [2b]. However, as we show in this paper, one can always find a complex signal

1. We note that although the quadrature method does not generally yield an analytic signal, in many cases the signal obtained is approximately analytic; this is so when the spectrum of the complex signal obtained via the quadrature method has very little energy for frequencies $\omega < 0$ [12], [13].

2. Unless otherwise noted, integration spans $(-\infty, \infty)$.



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for which the derivative of the phase equals the conditional mean frequency of a positive TFD. Furthermore, the derivative of the phase of this signal is a reasonable definition of instantaneous frequency. For example, the distribution $P(t, \omega) = (1/\pi) e^{-\alpha t^2 - (\omega - \omega_0 - \beta t)^2/\alpha}$ of the signal $z(t) = (\alpha/\pi)^{1/4} e^{-\alpha t^2/2 + j(\omega_0 t + \beta t^2/2)}$ satisfies eq. (2) [3]. Likewise, the distribution $P(t, \omega) = |z(t)|^2 |Z(\omega)|^2$, where $Z(\omega)$ is the Fourier transform of signal $z(t)$, satisfies eq. (2) for $z(t) = A(t) e^{j\omega_0 t}$ [6].[3] Neither of these signals is necessarily analytic (the spectrum is not necessarily zero for $\omega < 0$), yet in each case the derivative of the phase is reasonable and sensible as the "instantaneous frequency" of the signal. The TFDs are sound as well, in that they are nonnegative and yield the correct marginal densities. They belong to the class of positive distributions introduced by Cohen with Zaparovanny and Posch [6], [7].[4]

We explore these issues further in the remainder of the paper.

## Examples and Results

Consider the two-tone signal

$$s(t) = A_1 e^{j\omega_1 t} + A_2 e^{j\omega_2 t} = A(t) e^{j\varphi(t)} \tag{3}$$

where $A_1, A_2$ are real constants. Note that this signal is analytic if $\omega_1, \omega_2 > 0$. The derivative of the phase, which we take as the instantaneous frequency of the signal, is readily calculated as [4], [5]

$$\dot{\varphi}(t) = \frac{1}{2}(\omega_2 + \omega_1) + \frac{1}{2}(\omega_2 - \omega_1)\frac{A_2^2 - A_1^2}{A^2(t)} \tag{4}$$

where

$$A^2(t) = A_1^2 + A_2^2 + 2A_1 A_2 \cos((\omega_2 - \omega_1)t) \tag{5}$$

The only case for which the instantaneous frequency does not extend beyond the tones at $\omega_1$ and $\omega_2$ is when the tones are of equal strength, i.e., for $|A_1| = |A_2|$, as we now prove. For the instantaneous frequency $\dot{\varphi}(t)$ to remain bounded by $\omega_1$ and $\omega_2$, we require

$$\left|\frac{A_2^2 - A_1^2}{A_1^2 + A_2^2 + 2A_1 A_2 \cos((\omega_2 - \omega_1)t)}\right| \le 1 \tag{6}$$

or equivalently

$$A_1^2 + A_1 A_2 \cos((\omega_2 - \omega_1)t) \ge 0 \quad \text{and}$$
$$A_2^2 + A_1 A_2 \cos((\omega_2 - \omega_1)t) \ge 0 . \tag{7}$$

The cases $A_1 > 0, A_2 > 0$, and $A_1 < 0, A_2 < 0$ yield

$$\frac{A_1}{A_2} \ge -\cos((\omega_2 - \omega_1)t) \quad \text{and}$$
$$\frac{A_2}{A_1} \ge -\cos((\omega_2 - \omega_1)t) \tag{8}$$

The cases $A_1 > 0, A_2 < 0$ and $A_1 < 0, A_2 > 0$ result in

$$\frac{A_1}{A_2} \le -\cos((\omega_2 - \omega_1)t) \quad \text{and}$$
$$\frac{A_2}{A_1} \le -\cos((\omega_2 - \omega_1)t) \tag{9}$$

The only solution to eqs. (8) and (9) is $|A_1| = |A_2|$. For unequal strength tones, $\dot{\varphi}(t)$ is erratic and not amenable to interpretation as the average frequency at each time; see the dotted line plotted in figure 2b, for example.

Now let's consider three different TFDs of this signal. Shown in figure 1 are a time-conditional spectrogram, a time-conditional Choi-Williams distribution [1], and a time-conditional positive distribution, obtained via the method of [9], for the two-tone signal with $|A_1| \ne |A_2|$. We plot the conditional distribution in each case because that is the distribution from which one calculates the conditional mean frequency — see eq. (1). As is well known, the spectrogram is nonnegative, but does not yield the correct marginals; the Choi-Williams TFD yields the correct marginals, but goes negative; and the positive TFD is nonnegative and yields the correct marginals.

Of the three distributions, the positive TFD is arguably the best measure of the time-frequency energy density of this particular signal. Unlike the other two, it indicates that the signal was a sum of two constant frequency tones, it is nonnegative (as any energy density should be), and the joint density yields the correct marginals (as any joint density should). Furthermore, for $|A_1| = |A_2|$, the conditional mean frequency of the distribution equals the derivative of the phase (see figure 2a), which in this case is the average of the two tones. The TFD in figure 1(c), which is the distribution for $|A_1| \ne |A_2|$, likewise yields a weighted average of the two tones for $\langle\omega\rangle_t$, consistent with the result for $|A_1| = |A_2|$. See figure 2a,b. That this result does not equal $\dot{\varphi}(t)$ in this case should be no cause for alarm, as we already know from above that $\dot{\varphi}(t)$ behaves oddly for $|A_1| \ne |A_2|$ (see figure 2b). Why insist, then, that an otherwise perfectly reasonable distribution yield this result for the conditional mean frequency?

No positive TFD, including the one shown in figure 1c, can ever yield $\dot{\varphi}(t)$ for $\langle\omega\rangle_t$ if $\dot{\varphi}(t)$ extends beyond the bandwidth of the signal, because such a result conflicts with positivity and the frequency marginal [2b], [4], [9]. To see this, consider a signal whose spectrum is zero outside the interval $\omega_l < \omega < \omega_u$, but whose instantaneous frequency is not zero outside this interval. The conditional

3. Unit-energy signals are assumed. Amplitudes $A(t)$ are real.
4. Throughout the paper, by "positive distribution" or "positive TFD," we mean the distributions of Cohen, Posch and Zaparovanny. Thus, a spectrogram, although positive, is not a positive distribution in this sense because it does not satisfy the marginals.

average over frequency of a positive TFD will equal this instantaneous frequency only if the distribution has non-zero mass centered about the instantaneous frequency curve in the time-frequency plane. But such a requirement conflicts with the frequency marginal, which requires that the positive TFD be zero outside the interval $\omega_l < \omega < \omega_u$.

There are many signals for which a positive TFD yields $\dot{\varphi}(t)$ for $\langle\omega\rangle_t$, as shown in figure 2 for[5]

$$z(t) = e^{j2\pi 6t^2} \tag{10}$$

$$z(t) = e^{j2\pi t^3} \tag{11}$$

$$z(t) = e^{j2\pi 0.5t^4} \tag{12}$$

$$z(t) = e^{j2\pi(8t^2 + 12t^3 - 5.5t^4)}. \tag{13}$$

Furthermore, given a signal and a positive TFD of the signal, it is simple to find a complex signal such that eq. (2) holds for the positive TFD: calculate $\langle\omega\rangle_t$ from a positive TFD of the given signal,[6] then integrate it to get a phase

$$\varphi(t) = \int_{-\infty}^{t} \langle\omega\rangle_\tau d\tau. \tag{14}$$

Take the amplitude as

$$A(t) = s(t)/\cos(\varphi(t)) \tag{15}$$

to obtain the complex signal. By design, the real part of this signal equals the real part of the given signal $s(t)$, and the derivative of the phase of this signal equals the conditional mean frequency of the positive TFD of the given signal.

## Discussion and Conclusion

As commonly defined, instantaneous frequency is the derivative of the phase of a complex signal. Nature gives us real signals; going from a real signal to a complex signal is a one-to-many mapping, in that there are an infinite number of complex signals whose real part is the given real signal. In other words, there are an infinite number of possibilities for the amplitude and phase of the complex signal.

Although Gabor proposed a method for unambiguously *defining* the amplitude and phase from the given real signal, namely via the analytic signal, the derivative of the phase of the analytic signal often behaves erratically and yields results that are difficult to interpret as the "average frequency at each time." Sometimes the analytic signal gives sensible results, consistent with positivity and the marginals (e.g., figure 2a), but sometimes it doesn't (figure 2b).

The results presented here suggest that an alternative approach to defining instantaneous frequency may be fruitful. Rather than letting $\dot{\varphi}(t)$ dictate what $\langle\omega\rangle_t$ should be, the converse approach should be taken. That is, the approach has traditionally been to decide beforehand on a particular complex signal, and then insist that the TFD yield $\dot{\varphi}(t)$ for $\langle\omega\rangle_t$. Instead, we suggest that $\langle\omega\rangle_t$ be calculated from a sound TFD, namely one that, at the very least, is nonnegative and yields the correct marginals. That then dictates the phase of a particular complex signal, per eq. (14). As shown, using this approach one can always find a complex signal for which eq. (2) holds for the positive TFD. Of course, then the issue is one of finding the sound TFD, but that has *always* been one of the fundamental goals of time-frequency analysis. It remains an unsolved problem. When a complete theory for TFDs is found, the definition of instantaneous frequency will follow naturally.

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5. For computational purposes, the signals were discretized by taking $t = nT$, where $n$ is an integer and $T$ is a sampling period. The instantaneous frequency was calculated as $\dot{\varphi}(t)|_{t=nT}$, and the conditional mean frequency was computed via the method of Lovell *et al.* [10].

6. If the given signal is real, calculate $\langle\omega\rangle_t$ as a one-sided conditional average to obtain a conditional mean frequency different from zero; i.e., $\langle\omega\rangle_t = 2\int_0^\infty \omega P(\omega|t)\,d\omega = \langle\omega\rangle_t^+$. Or, use the analytic signal and its positive TFD, and proceed as outlined.

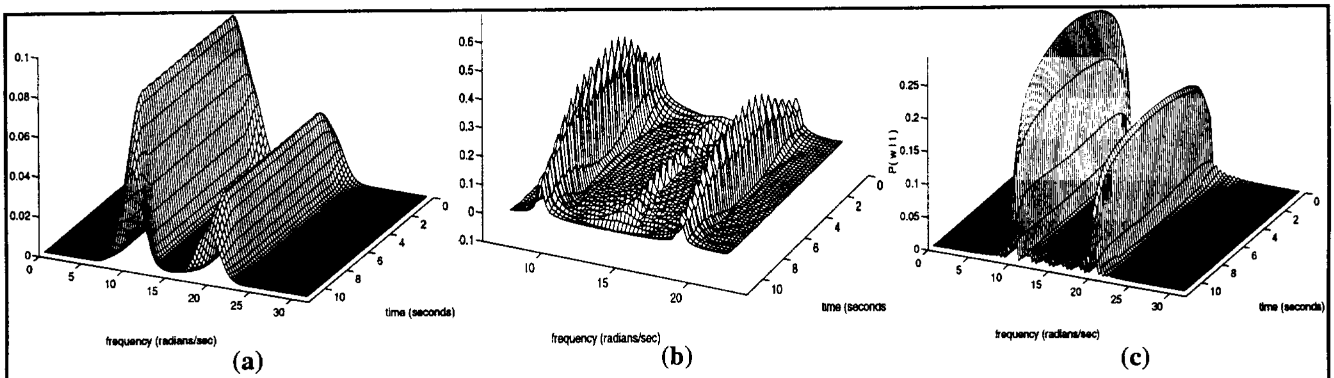

Figure 1: The time-conditional distributions of a two-tone signal (eq. (3)) for (a) a spectrogram, (b) a Choi-Williams distribution and (c) a Cohen-Posch (i.e., positive) distribution. The conditional mean frequency is a weighted sum over frequency of the conditional distribution — see eq. (1). The amplitudes of the tones are $A_1$=1.2, $A_2$=1.

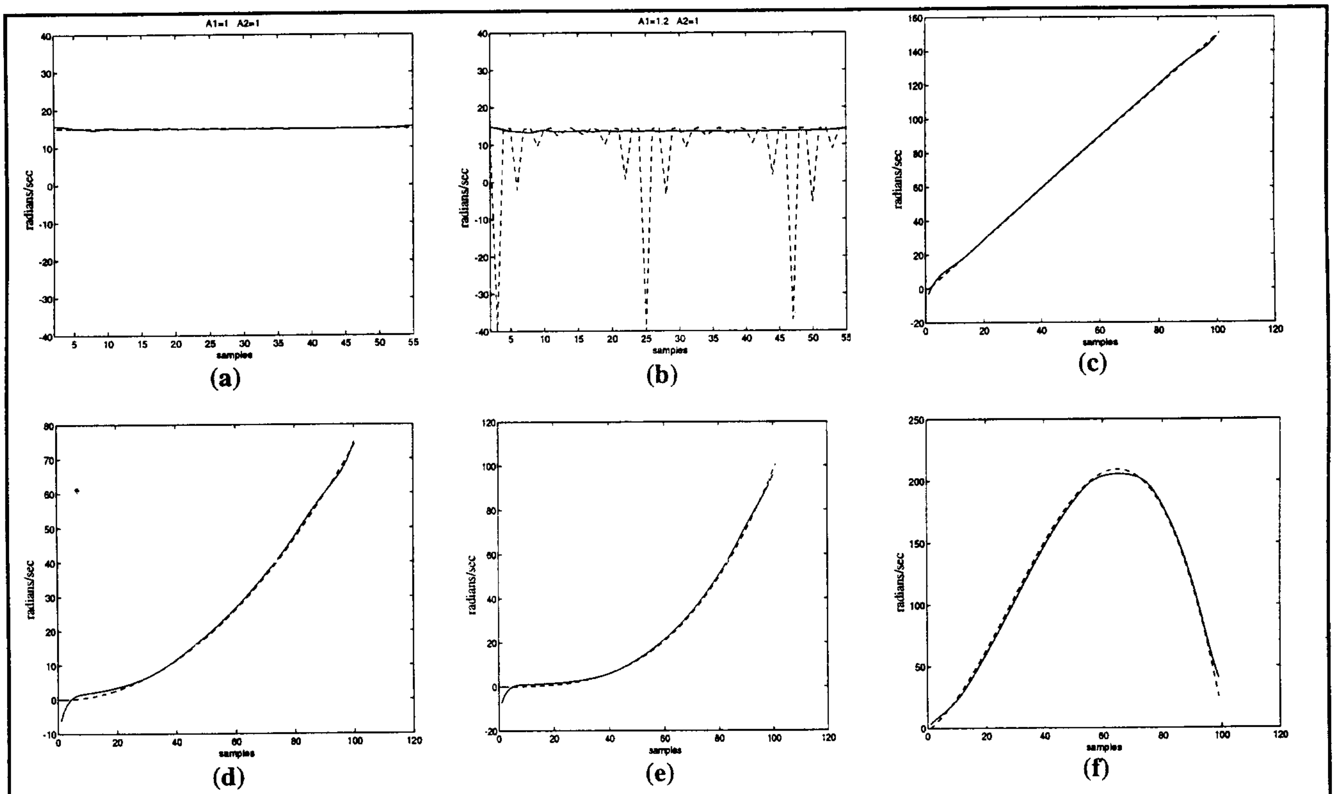

Figure 2: The conditional mean frequency calculated from a positive TFD (solid line) and the instantaneous frequency (dashed line) for several different signals. The signals are (a) two tones of equal amplitude (eq. (3)); (b) two tones of different amplitude ($A_1$=1.2, $A_2$=1) (eq. (3)) (c) linear FM (eq. (10)); (d) quadratic FM (eq. (11)); (e) cubic FM (eq. (12)); polynomial FM (eq. (13)). Except for slight computational edge effects, the conditional mean frequency equals the instantaneous frequency in (a), (c)-(e), and shows close agreement in (f). For the signal where the two differ, namely (b), the instantaneous frequency extends outside the spectral bandwidth of the signal; the conditional mean frequency, on the other hand, does not and it is consistent with the interpretation of "average frequency at each time." For the two-tone signal in (a) and (b), the case of equal magnitude tones is the only one for which the instantaneous frequency does not extend outside the bandwidth of the signal. The effect of changing the amplitudes of the tones has a consistent effect on the conditional mean frequency of the positive TFD, in that it remains a weighted average of the frequencies present at each time, moving from the average of the tones when the amplitudes are equal in (a) towards the tone of larger magnitude in (b). The instantaneous frequency, on the other hand, is quite erratic as one tone becomes stronger than the other. One can always find a complex signal for which eq. (2) holds for any positive TFD — see text for details.