QUADRALINEAR TIME-FREQUENCY REPRESENTATIONS

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ABSTRACT

In time-frequency analysis there are four bilinear domains commonly used: time-frequency, ambiguity, temporal auto-correlation, and spectral auto-correlation. This paper introduces four new domains that are quadralinear in the signal. These four domains are each a function of three of the four variables used in the bilinear domains. Properties of these quantities are developed and an application is given for the design of adaptive time-varying kernels.

1. INTRODUCTION

Two linear domains that are useful for representing a signal are the time domain and the frequency domain. The time-frequency functions extend these two linear domains to four bilinear domains. A signal can be represented as a Wigner distribution (WD), an ambiguity function (AF), a temporal auto-correlation function (TACF), and a spectral auto-correlation function (SACF). Each of these functions depend on two of the four variables; time, frequency, lag, and Doppler[1], and the functions are all related to each other by Fourier transforms. A summary of these relationships is shown in Figure 1 where the direction of the arrow represents the direction of the forward transform. Each function has properties that make it more useful for a particular application. For example, the WD is useful for representing the time-frequency content of the signal, while the AF is useful for designing kernels that eliminate crossterms [1].

This paper introduces four quadralinear domains that are functions of three of the four variables mentioned above. The quadralinear functions are derived from the bilinear functions in the same way that the bilinear functions are derived from the linear functions. The quadralinear functions are described below with an application to adaptive time-varying kernel design. An octalinear function is also introduced that is a function of all four variables.

The functions defined here bear resemblance to the Wigner higher order moment spectra defined by Fonollosa and Nikias [2], the polynomial Wigner distribution defined by Boashash and O'Shea [3], and the L-Wigner distribution defined by Stanković [4].

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[1] Lag denotes time lag while Doppler denotes frequency lag.

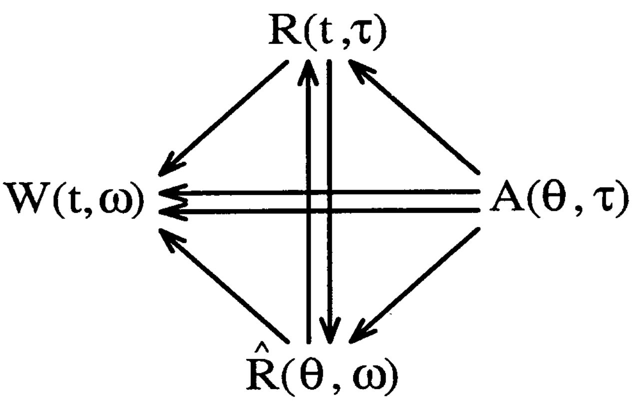

Figure 1: Domain relationships.

2. QUADRALINEAR TIME-FREQUENCY FUNCTIONS

This section derives the four quadralinear functions. In this section, $x(t)$ and $X(\omega)$ will denote an arbitrary time signal and its Fourier transform respectively. This section will apply what we will call a Wigner mapping. This is the operation that is performed on a time signal to obtain a Wigner distribution, however, here it will be applied to one variable of a multi-variable function. For example, if M is the Wigner mapping of N then:

$$M(\alpha,\beta,\gamma) = \int N(\alpha+\frac{\delta}{2},\gamma)N^*(\alpha-\frac{\delta}{2},\gamma)\,e^{-j\delta\beta}\,d\delta$$

An inverse Wigner mapping will be the same as the Wigner mapping with the Fourier transform replaced with an inverse Fourier transform. For example, if M is the inverse Wigner mapping of N then:

$$M(\alpha,\beta,\gamma) = \frac{1}{2\pi}\int N(\alpha+\frac{\delta}{2},\gamma)N^*(\alpha-\frac{\delta}{2},\gamma)\,e^{j\delta\beta}\,d\delta$$

Neither mapping is invertible; one corresponds to a Fourier transform, while the other corresponds to an inverse Fourier transform.

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2.1. Example Derivation

The four different quadralinear functions are all derived by using the same method. This section details the derivation of the time-lag-Doppler function (TLDF) and can be applied to derive the other three quadralinear functions.

Normally, to compute an AF one starts with the TACF and performs an inverse Fourier transform with respect to time.

$$A(\theta,\tau) = \frac{1}{2\pi}\int R(t,\tau)\, e^{jt\theta}\, dt \qquad (1)$$

To generate a function that depends on time, lag, and Doppler, replace the inverse Fourier transform in equation 1 with an inverse Wigner mapping. The inverse Fourier transform takes one from the t domain to the θ domain, while the inverse Wigner mapping takes one from the t domain to a combined t-θ domain. The TLDF is defined as

$$TLDF(t,\tau,\theta) = \frac{1}{2\pi}\int R(t+\frac{\delta}{2},\tau)R^*(t-\frac{\delta}{2},\tau)\, e^{j\delta\theta}\, d\delta \qquad (2)$$

The same quantity can be derived by going in the opposite direction. The TACF can be represented in terms of the AF.

$$R(t,\tau) = \int A(\theta,\tau)\, e^{-jt\theta}\, d\theta \qquad (3)$$

As above, replace the Fourier transform in equation 3 with a Wigner mapping to arrive at a function that depends on time, lag, and Doppler.

$$TLDF(t,\tau,\theta) = \int A(\theta+\frac{\gamma}{2},\tau)A^*(\theta-\frac{\gamma}{2},\tau)\, e^{-j\gamma t}\, d\gamma \qquad (4)$$

To show that equation 2 and equation 4 are indeed the same, replace the AF in equation 4 with its definition in equation 1 and perform a change of variables.

2.2. General Properties

The four quadralinear functions have some basic properties in common. They are all real, and as the name implies, a fourth order function of the signal.

The above function can be interpreted as a time-varying AF or a Doppler-varying TACF (if there was a bilinear function of time and Doppler then it could also be considered a lag-varying time-Doppler function). To avoid this ambiguity they will be given unimaginative names such as the time-lag-Doppler function.

Each of the quadralinear functions can be derived in two ways. One replaces a Fourier transform with a Wigner mapping, and the other replaces an inverse Fourier transform with an inverse Wigner mapping. Since the two derivations have the same variables, it would be nice if they were actually the same. This is indeed the case and can be shown as mentioned above.

Unlike the bilinear functions, the quadralinear functions are not related to each other by Fourier transforms. This is easily seen after the properties of the individual functions are developed below.

2.3. The Time-Lag-Doppler Function

The time-lag-Doppler function derived above is rewritten here as a function of the signal.

$$TLDF(t,\tau,\theta) = \frac{1}{2\pi}\int x(t+\frac{\tau}{2}+\frac{\delta}{2})x^*(t+\frac{\tau}{2}-\frac{\delta}{2})\, x^*(t-\frac{\tau}{2}+\frac{\delta}{2})x(t-\frac{\tau}{2}-\frac{\delta}{2})\, e^{j\delta\theta}\, d\delta$$

Some properties of the TLDF:

- the TLDF of $x(t)$ and $x(t)e^{ja}e^{jbt}$ are the same; thus, the TLDF is independent of phase shifts and frequency shifts,
- there are no crossterms in the TLDF of a quadratic chirp.

The TLDF has the effect of reducing a quadratic chirp to a complex exponential, hence there are no crossterms. If this function is considered to be a time-varying AF then one application is in the design of time-varying kernels. This is discussed in more detail in a later section.

2.4. The Frequency-Lag-Doppler Function

The frequency-lag-Doppler function (FLDF) can be derived by expressing the AF in terms of the SACF and applying the above method. The derivation is skipped and the FLDF is shown below.

$$FLDF(\omega,\tau,\theta) = \frac{1}{2\pi}\int X(\omega+\frac{\theta}{2}+\frac{\gamma}{2})X^*(\omega+\frac{\theta}{2}-\frac{\gamma}{2})\, X^*(\omega-\frac{\theta}{2}+\frac{\gamma}{2})X(\omega-\frac{\theta}{2}-\frac{\gamma}{2})\, e^{j\gamma\tau}\, d\gamma$$

The properties of the FLDF are analogous to those of the TLDF above.

- the FLDF of $X(\omega)$ and $X(\omega)e^{ja}e^{jb\omega}$ are the same; thus, the FLDF is independent of phase shifts and time shifts,
- there are no crossterms in the FLDF of a time signal with a quadratic group delay.

If the FLDF is considered to be a frequency-varying AF then one application is in the design of frequency-varying kernels.

2.5. The Time-Frequency-Lag Function

The time-frequency-lag function (TFLF) can be derived by expressing the WD in terms of the TACF and applying the above method. The derivation is skipped and the TFLF is shown below.

$$TFLF(t,\omega,\tau) = \int x(t+\frac{\tau}{2}+\frac{\delta}{4})x^*(t-\frac{\tau}{2}-\frac{\delta}{4})\, x^*(t+\frac{\tau}{2}-\frac{\delta}{4})x(t-\frac{\tau}{2}+\frac{\delta}{4})\, e^{-j\delta\omega}\, d\delta$$

This function looks similar to the others but the properties are quite different:

- the TFLF *depends* on both time shifts and frequency shifts,
- the TFLF of a linear chirp is equal to the WD of a linear chirp,
- there *are* crossterms in the TFLF of a quadratic chirp and also a signal with quadratic group delay.

If we set $\tau = 0$ in the TFLF we obtain the following.

$$TFLF(t,\omega,0) = \int x(t+\frac{\delta}{4})^2 x^*(t-\frac{\delta}{4})^2 e^{-j\delta\omega}\, d\delta$$

Comparing this to the Wigner distribution, an intuitive generalization to higher powers results in the L-Wigner distribution recently defined by Stanković [4].

$$LWD_L(t,\omega) = \int x(t+\frac{\tau}{2L})^L x^*(t-\frac{\tau}{2L})^L e^{-j\omega\tau}\, d\tau$$

For $L = 1$ this reduces to the Wigner distribution and for $L = 2$ this reduces to TFLF with $\tau = 0$.

2.6. The Time-Frequency-Doppler Function

The time-frequency-Doppler function (TFDF) can be derived by expressing the WD in terms of the SACF and applying the above method. The derivation is skipped and the TFDF is shown below.

$$TFDF(t,\omega,\theta) = \int X(\omega+\frac{\theta}{2}+\frac{\gamma}{4})X^*(\omega-\frac{\theta}{2}-\frac{\gamma}{4}) X^*(\omega+\frac{\theta}{2}-\frac{\gamma}{4})X(\omega-\frac{\theta}{2}+\frac{\gamma}{4})\, e^{-j\gamma t}\, d\gamma$$

This function has the same properties as listed above for the TFLF. If we set $\theta = 0$ in the TFDF and generalize to higher powers we obtain what could be considered to be a dual form of the L-Wigner distribution.

$$LWD_L(t,\omega) = \int X(\omega+\frac{\theta}{2L})^L X^*(\omega-\frac{\theta}{2L})^L e^{-j\theta t}\, d\theta$$

For $L = 1$ this reduces to the Wigner distribution and for $L = 2$ this reduces to TFDF with $\theta = 0$.

3. EXAMPLE

In this section we show some examples of the TLDF for a signal whose instantaneous frequency is sinusoidal. The example signal used is a Hanning windowed version of:

$$x(t) = e^{j(\frac{2\pi t}{4} - 8\cos(\frac{2\pi t}{64}))}$$

and its Wigner distribution is shown in Figure 2. The TLDF is calculated at times 48 and 64 and is also shown in Figure 2. Since we are displaying the function at fixed times it is more intuitive to think of this quantity as a time-varying AF.

The TLDF at time 48 is similar to the AF of a complex exponential. This makes sense, since at time 48 the tangent of the instantaneous frequency of the example signal is a horizontal line. In other words the example signal is locally "like" a complex exponential.

The TLDF at time 64 is similar to the AF of an increasing linear chirp. Again this makes sense since at time 64 the tangent of the instantaneous frequency of the example signal is a line with positive slope. In other words the example signal is locally "like" an increasing linear chirp. In between these two times, the TLDF gradually progresses as expected.

4. TIME-VARYING KERNEL DESIGN

An application of the TLDF and FLDF is the design of time-varying kernels and frequency-varying kernels respectively. Baraniuk and Jones [5, 6] have designed algorithms that derive adaptive time-invariant kernels based on the AF. They then extended this to derive time-varying kernels by using a short-time ambiguity function (STAF) [7]. A slightly different time-varying kernel can be constructed by replacing the STAF with a TLDF and using the same adaptive algorithm.

The relationship between the TLDF and the STAF is similar to that between the Wigner distribution and the spectrogram. Applying an appropriate kernel to the construction of the TLDF could result in a representation of the "instantaneous" AF that is better than the STAF(in the same manner that members of Cohen's class can give better results than the spectrogram). Now, however, one needs to design a kernel to construct a TLDF that is used to design a time-varying adaptive kernel. This two level kernel design requirement is probably too complicated to be a reasonable solution.

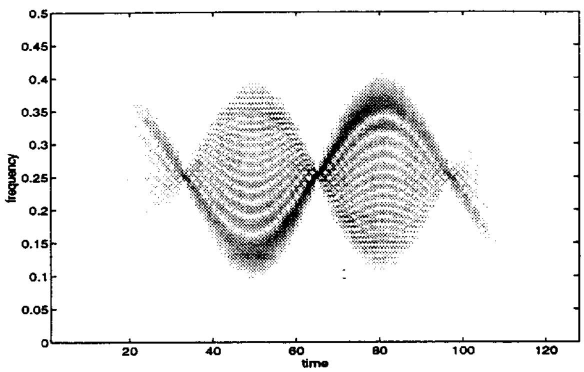
Windowing techniques were used to attenuate crossterms in the TLDF. With this addition, representations produced from the two different time-varying kernels (designed from the TLDF and the STAF) were very similar. However, due to computational advantages, the STAF algorithm would be preferred.

5. A TIME-FREQUENCY-LAG-DOPPLER FUNCTION

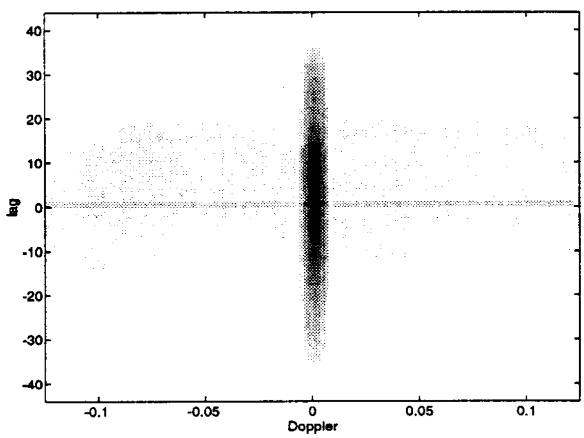
It is also possible to construct a function of all four time-frequency variables. One way to do this would be to write the SACF in terms of the TACF and replace both Fourier transforms with Wigner mappings. There are also other ways to construct this function, but unlike the quadralinear functions, they do not seem to be equivalent. An application would be in the design of time and frequency varying kernels.

6. CONCLUSION

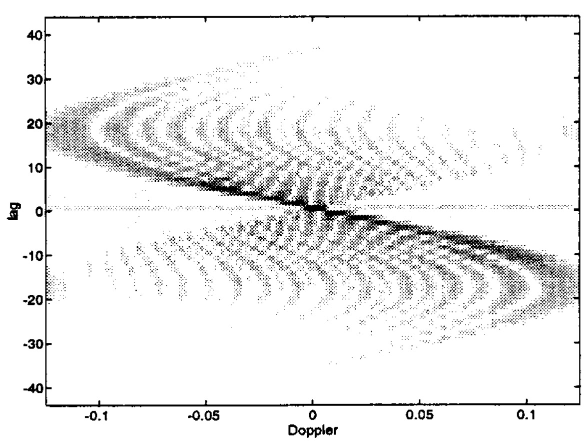
Higher order representations previously defined [2, 3, 4] are functions of one or more time and frequency variables. This paper introduces quadralinear representations that are functions of three different variables, at least one of which is lag or Doppler. The quadralinear representations suggest



(a)



(b)



(c)

Figure 2: (a) Wigner Distribution of Example Signal, (b) TLDF at time 48, (c) TLDF at time 64.

an alternative motivation for using higher order representations, and may provide further insight into the use of higher order functions for the purpose of signal representation. An example shows that the TLDF satisfies what we would intuitively expect from an "instantaneous" AF. The quadralinear functions can be applied to the design of time and/or frequency varying kernels, though to gain advantage over current methods requires unreasonably complex solutions.

7. REFERENCES

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