RÉNYI INFORMATION AND SIGNAL-DEPENDENT OPTIMAL KERNEL DESIGN

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ABSTRACT

The Rényi uncertainty measure [2][4] has been proposed to be a measurement of complexity of signals. We further suggest that it can be used to evaluate the performance of different time-frequency distributions(TFDs). We provide two schemes of normalization in calculating the Rényi uncertainty measure. For the first one, TFDs are normalized by their energy, while for the second one,normalized with their volume. The behavior of the Rényi uncertainty measure under several situations is studied. A signal-dependent algorithm is developed to achieve TFDs with a minimal uncertainty measure.

1. INTRODUCTION

The lack of a quantitative criteria to evaluate the performance of different kernels and guidelines to design effective kernels is always an obstacle for promoting the usage of time- frequency analysis. Our goal is to tackle this problem by utilizing some useful aspects of the generalized Rényi uncertainty measure.

The discrete-type generalized Rényi information is defined as:

$$R_\alpha = \frac{1}{1-\alpha}\log_2\{\sum_{l=-L}^{L}\sum_{k=-K}^{K}[C_s(l,k)]^\alpha\} \qquad (1)$$

where $C_s(l,k)$ is a TFD of Cohen's class. Flandrin, Baraniuk, and Michel [2] have established several important properties of this Rényi uncertainty measure. We claim that better TFDs are those with smaller uncertainty measure. It is helpful to an analogy between TFDs and two-dimensional probability distribution functions (pdf), therefore, before one applies equation (1) to a TFD $C_s(l,k)$, there is a need to "normalize" it in some way. We suggest two different schemes of normalization. The first one is to normalize $C_s(l,k)$ with respect to its energy, i.e., $\sum\sum C_s(l,k)$, while the second one is to normalize it with respect to its volume, i.e., $\sum\sum|C_s(l,k)|$.

We developed a recursive algorithm to compute signal-dependent product kernels which can generate minimum-uncertainty-measure distributions for given signals. For the first normalization scheme, the Wigner distribution (WD) is found to be optimal or near-to-optimal under certain constraints. If the second scheme is used, our program can generate minimum-uncertainty product kernels which are very effective at suppressing cross terms and maintaining high resolution.

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2. THE PROPERTIES OF RÉNYI INFORMATION

The most significant difference between a TFD and a pdf is that a pdf is always non-negative by definition, but a TFD of Cohen's class can take on negative values.

Case 1. R_2 with energy normalized. The WD can be shown to have the minimum Rényi uncertainty measure. By Parseval's theorem, this case is equivalent to maximizing the following summation:

$$\frac{1}{4\pi^2}\sum_{m=-L}^{L}\sum_{n=-K}^{K}|A(m,n)\phi(m,n)|^2 \qquad (2)$$

where $A(m,n)$ is the ambiguity function and the kernel is under the constraint $|\phi(m,n)| \leq 1$. It is obvious that the WD will maximize this, thus minimizing the uncertainty measure. One may note that the work in [1] is equivalent to minimizing R_2 with some constraints posed as a parameter which determines the extent to how close the resulting kernel should be moved to the kernel of the WD.

Case2. R_2 with volume normalized. It is difficult to analyze the effect of the normalization with volume, but we can construct some figurative arguments.

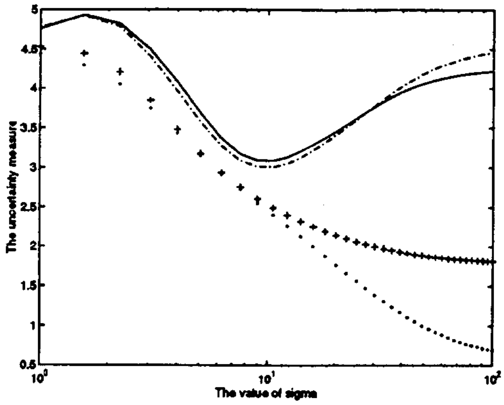

Figure 1: σ v.s. Rényi Uncertainty Measure: case 1 (...); case 2 (_); case 3 (+++); case 4 (_._)

Now, both auto terms and cross terms contribute to R_2 even when the cross terms do not overlap with the auto terms. Under this condition, the sharpness (resolution) and the relative magnitude of auto and cross terms will be two major factors affecting the uncertainty measure.

Case 3. R_3 with energy normalized. Here we consider only the simple situation, e.g., when the auto terms and cross terms in a TFD don't overlap. It can be shown that cross terms will have no contribution in odd-order Rényi uncertainty measure as long as auto terms are separated well enough. Under this situation, it can be argued that the WD will be optimal or near-to-optimal. In general, the effect of multiplying a kernel function in the ambiguity domain is to "smear" the auto terms in the time-frequency domain, and thus increase the uncertainty measure.

Case4. R_3 with volume normalized. Again, we shall use some figurative arguments. The cross terms still don't contribute to R_3 under the same assumptions of case 3, but now the total volume is affected by cross terms via the operation of normalization. As a result, the TFDs with smaller cross terms (measured in volume) will have a lower uncertainty measure than that of TFDs which have larger cross terms and the same resolution. In this sense, our algorithm seeks a balance between the trade-off of suppressing cross terms and enhancing resolution.

The above discussion is best illustrated by figure 1. The signal used here is a four-tone frequency hop and the kernel used is the exponential type, i.e., $\phi(m,n) = \exp(-mn/\sigma^2)$. We vary the value of σ and compute the uncertainty measure for the above four cases. In case 2 and 4, we observe that the minima occur around $\sigma = 10$, while in case 1 and 3, the larger σ is (the closer TFD is to the WD), the lower uncertainty measure. Notice that the uncertainty of case 1 approaches zero as the distribution approaches the WD, which means the WD of it will have the same uncertainty as the WD of a single Gabor logon does. It has also been pointed out in [2] as a reason why the second order Rényi uncertainty measure is not appropriate for measuring complexity in TFDs.

It is much more difficult to analyze the behavior of Rényi uncertainty measure when cross terms and auto terms overlap one another. In [2] the authors pointed out that Rényi information has great phase sensitivity. This is also noted in [4] in terms of undue peaking for certain Gabor logon spacings. This can be illustrated by the example of varying the distance of two Gabor logons which have $\pi/2$ phase difference (please see figure 2). The logons are in the form of $\exp(-t^2/40)\exp(jt)$.

It is not only a matter of phase sensitivity but also a matter of how to interpret a signal. The Rényi uncertainty seems to very well reflect the resolution of signals. If we observe the time marginal when the uncertainty measure hits the peak, we actually see two well-separated signals rather than a single component which we expected from two closely-placed Gabor logons as illustrated in figure 3. It is more logical to view the signal not simply as "the addition of two Gabor logons" under these circumstances. We are far from fully understanding the characteristics of the Rényi uncertainty measure applied to TFDs, but the measure does seem to consistently reflect what is intuitively considered to be "good" versus "bad" resolution.

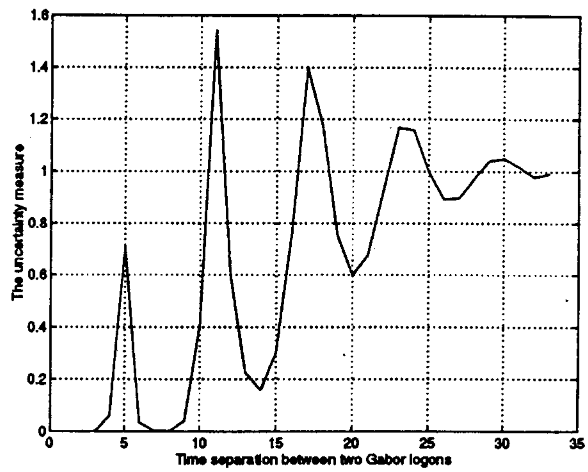

Figure 2: This figure illustrates the Rényi uncertainty measures of two Gabor logons separated by different time separations. Notice that the measure varies rapidly according to different time separations [2], but the uncertainty eventually increases by 1 bit.

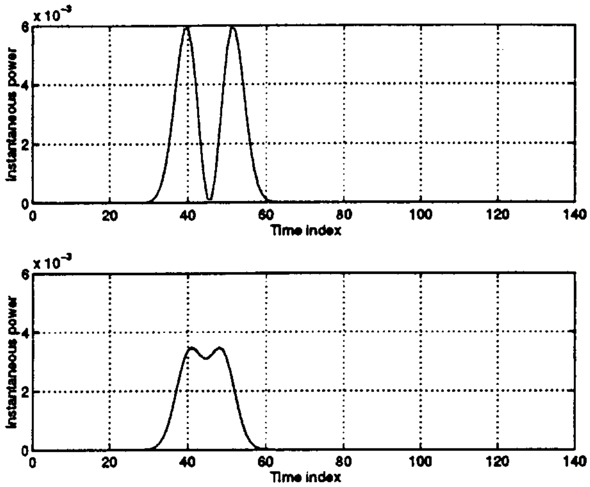

Figure 3: This figure shows the time marginals of the two Gabor logons when the time distance is 11 and 9 respectively.

3. DESIGNING ADAPTIVE KERNELS MINIMIZING UNCERTAINTY MEASURE

In this section we will describe the method of designing product kernels based on the method proposed by Jeong and Williams. The kernel function is derived from a real-valued primitive function $h(t)$ as:

$$\phi(\theta\tau) = H(\theta\tau) = \int h(t)e^{-j\theta\tau t}dt \qquad (3)$$

where $H(\theta\tau)$ is the Fourier transform of $h(t)$. The discrete version of it is:

$$\phi(m,n) = \sum_{u=-U}^{U} h(u)e^{-ju(\frac{\pi}{LK})mn} \qquad (4)$$

where $h(u)$ is the samples of $h(t)$ and $m = -L, \dots, L$; $n = -K, \dots, K$. L and K are the sizes of the signal's auto-correlation matrix. Now we apply the steepest gradient method to adjust the primitive function $h(u)$ under the constraint $\sum h(u) = 1$:

$$h(u)_{k+1} = h(u)_k - \beta\frac{\partial R_\alpha}{\partial h(u)_k} \qquad (5)$$

where β is the step size for each iteration. In the rest of the section we will consider the case of 3rd order Rényi uncertainty measure with volume normalized. The partial derivative part in equation (5) can be calculated by the following: Define

$$I_3 = \sum_{l=-L}^{L}\sum_{k=-K}^{K}\left[\frac{C_s(l,k)}{\sum\sum|C_s|}\right]^3$$

we have

$$\begin{aligned}\frac{\partial R_3}{\partial h(u)} &= \frac{-1}{2I_3\ln 2}\frac{\partial I_3}{\partial h(u)}\\ &= \frac{3}{2I_3\ln 2}\left\{\frac{[\sum\sum \mathrm{sgn}(C_s)C_{s,u}]\sum\sum(C_s)^3}{(\sum\sum|C_s|)^4}\right.\\ &\quad\left.-\frac{\sum\sum(C_s^2C_{s,u})}{(\sum\sum|C_s|)^3}\right\} \qquad (6)\end{aligned}$$

with sgn(•) being the sign function and $C_{s,u}(l,k)$ is the TFD generated by the kernel function

$$\phi_u(m,n) = \begin{cases} 1 & : \; u = 0 \\ 2\cos(u\frac{\pi}{LK}mn) & : \; u = 1,\dots,U \end{cases}$$

4. EXPERIMENTS

In this section we will give two examples by using the uncertainty measure of case 4.

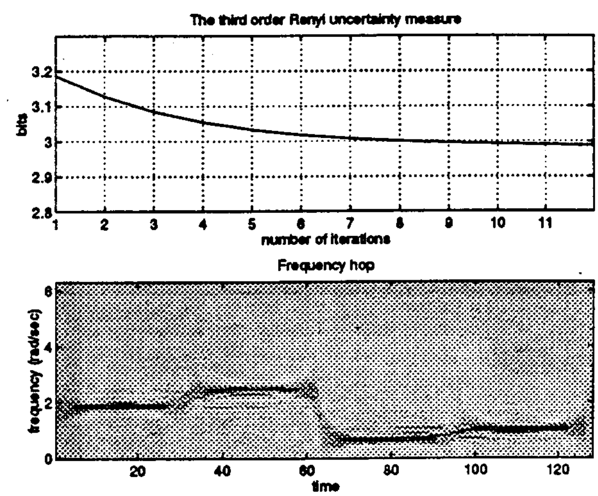

Figure 4: Four-tone frequency hop: the uncertainty measure and the TFD of the signal

Example 1. Four-tone frequency hop. The signal has four different frequencies: 1.9, 2.5, 0.7, and 1.1 (rad/sec). The uncertainty measure of case 4 is used. Figure 4 shows the uncertainty measure decreases with the iterations, and the resulted TFD of the signal. Figure 5 shows the primitive function and the kernel function in the ambiguity domain.

Example 2. Two parallel sinusoids. The signal $x(n) = \exp(j1.8n) + \exp(j2.6n)$. Figure 6 shows the TFD and the time marginal of the signal. The uncertainty measure of case 4 is used. Notice that the cross terms are almost totally eliminated and thus the TFD appears as two parallel dashed lines instead of two continuous lines in order to meet the constraint that the line integral of TFD along the frequency axis should be equal to the time marginal.

We have shown some results by using the 2nd order Rényi uncertainty measure in our previous paper [5]. We found that more stable results are obtained when the 3rd order Rényi uncertainty measure is used. In both cases, the primitive functions usually have an abrupt change (either a dip or a peak) at the center. It suggests that uncertainty measure is very sensitive to the weight of a WD component in the composition of RIDs.

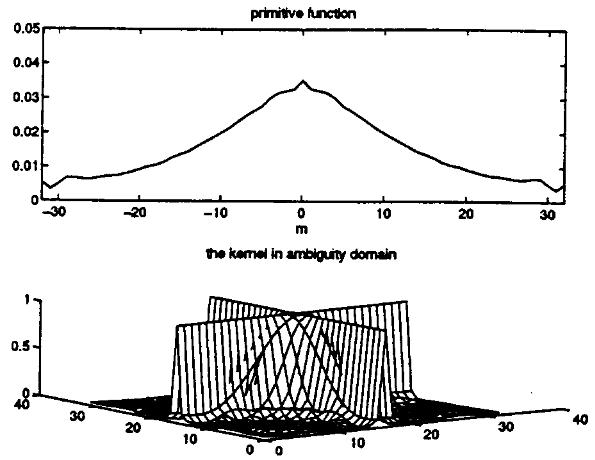

Figure 5: Four-tone frequency hop: the primitive function and the kernel function

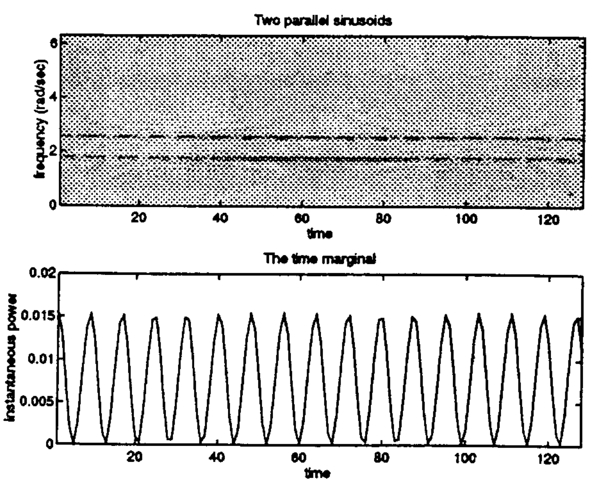

Figure 6: Two parallel sinusoids: the TFD and the time marginal.

5. CONCLUSION

The Wigner distribution has the highest resolution as well as the minimum Rényi uncertainty measure under the situation when the cross terms do not overlap with auto terms and when the first scheme of normalization is used. As a contrast, with the second scheme of normalization, RIDs have smaller uncertainty measure. We also developed an algorithm to generate a product kernel with minimum uncertainty measurement for given signals, and the result is generally satisfactory, often exhibiting much reduced cross terms.

6. REFERENCES

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