FAST MINIMUM VARIANCE RESAMPLING

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ABSTRACT

A novel method is introduced for resampling irregularly sampled data in the presence of noise. The estimator is minimum variance(MV) and minimum mean square error, under Gaussian assumptions, and well-conditioned in general. The Shannon-Whittaker sampling theorem is generalized to use raised cosine pulses as basis functions. It is shown that this generalization permits fast estimation with $O(N)$ computational requirements for mildly oversampled signals (bandwidth less than $0.9B_N$, where B_N is the Nyquist bandwidth of the resampled data). Also, some extensions of the inverse estimator and its error characteristics are discussed.

1. Introduction

The Shannon-Whittaker sampling theorem gives a formula and conditions for recovering a continuous-time signal $u(t)$ from its uniformly-spaced samples $u(nT)$. The formula is useful for resampling $u(t)$ at a different rate or even at non-uniformly spaced times t_i. What about recovering uniform samples $u(nT)$ from noisy non-uniform samples $u(t_i)$? This problem occurs in the reduction of data from multi-rate multiplexed data acquisition hardware, the tracking of speech movement with the University of Wisconsin X-ray Microbeam system being an example. We treat the recovery of non-uniform samples from uniform samples as a forward problem, and we formulate an inverse according to methods that arise in wavelet deconvolution, linear estimation, or statistical curve fitting. We greatly reduce the computational burden by substituting a raised cosine pulse for the sinc pulse in the Shannon-Whittaker formula. This limits the signal bandwidth to some fraction of the Nyquist bandwidth, and we can trade computational savings against how close this fraction is to unity.

Work completed at the University of Wisconsin-Madison. The authors are currently with MIT Lincoln Laboratory, Group 63, 244 Wood St., Lexington MA 02173-9108 and the University of Wisconsin-Madison, Electrical and Computer Engineering Dept. respectively.

2. Preliminary Theory

Consider a mixed analog-digital signal path with finite-energy analog input $u(t)$ with Fourier Transform(FT) $U(j\omega)$. The output, $y(t)$, is also analog with FT $Y(j\omega)$. The discussion used here is based on [5]. Start by sampling $u(t)$ as follows

$$\tilde{u}[k] = u(kT) \overset{DTFT}{\longleftrightarrow} \tilde{U}(e^{j\omega T}) = \frac{1}{T}\sum_n U(j\omega + jn\omega_0) \tag{1}$$

where $\omega_0 T = 2\pi$ and the Discrete-time Fourier Transform (DTFT) is defined as

$$f(k) = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} f(e^{j\omega T})e^{jk\omega T}\, d\omega \tag{2}$$

$$F(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} f(k)e^{-jk\omega T}. \tag{3}$$

Next, use reconstruction filter R with impulse response $r(t)$ so that

$$y(t) = \sum_k \tilde{u}[k]r(t-kT) \overset{FT}{\longleftrightarrow} Y(j\omega) = \tilde{U}(e^{j\omega T})R(j\omega). \tag{4}$$

If $U(j\omega)$ and $R(j\omega)$ are bandlimited to $|\omega| < \omega_0/2$, then the equivalent analog filter relationship is

$$Y(j\omega) = \frac{1}{T}R(j\omega)U(j\omega) \tag{5}$$

and

$$y(t) = \frac{1}{T}\sum_k u[kT]r(t-kT). \tag{6}$$

Note that the input samples $u[kT]$ are weights of the superposed impulse responses $r(t-kT)$. This is a generalization of the Shannon-Whittaker sampling theorem[5]. In that theorem, $R(j\omega) = T \cdot \mathrm{rect}(2\omega/\omega_0) \Longrightarrow r(t) = \mathrm{sinc}(\omega_0 t/2)$, and an interpolated sample $y(t_0)$ could depend on an infinite number of weighted $\mathrm{sinc}(t)$ functions.

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3. The Raised Cosine Pulse

One way to perform generalized resampling is to use the "raised cosine pulse" as a basis function. Instead of $r(t) = \text{sinc}(\omega_0 t/2)$, use

$$p(t) = \text{sinc}(\omega_0 t/2) \cdot \frac{\cos(\alpha\pi t/T)}{1-(4\alpha^2 t^2)/T^2}, \qquad (7)$$

where $0 \le \alpha \le 1$, as described in [4]. An important property of the raised cosine pulse is that its spectrum is unity for $\omega \le (1-\alpha)\omega_0/2$ and that the tails of $p(t)$ decay at least as fast as $1/t^3$. For sufficiently oversampled signals,

$$BW < (1-\alpha)\omega_0/2, \qquad (8)$$

this pulse has in no in-band distortion[4]. Also, the use of $p(t)$ results in efficient numerical methods as shown below.

4. The δ_n Method

Assume that $\mathbf{y}$ is a vector of noiseless exact observed samples, in M groups of n, as

$$\begin{aligned}\mathbf{y} \;=\; & [y(nT+\delta_1), y(nT+\delta_2), \ldots, y(nT+\delta_n), \\ & y(2nT+\delta_1), y(2nT+\delta_2), \ldots, y(MnT+\delta_n)]^T\end{aligned}$$

or

$$\mathbf{y} = [y(t_1), y(t_2), \ldots, y(t_{Mn})]^T$$

where T is the sampling interval defined above, and M and n are any positive integers. The goal is to estimate the resampled $\mathbf{u}$ where

$$\mathbf{u} = [y(nT), y((n+1)T), \ldots, y((Mn+n-1)T)]^T$$

or

$$\mathbf{u} = [y(t_1^*), y(t_2^*), \ldots, y(t_{Mn}^*)]^T.$$

Using (6) above, we form the linear equation

$$\mathbf{y} = \mathbf{H}\mathbf{u}$$

where

$$\mathbf{H}_{ij} = \frac{1}{T} p\left(t_i - t_j^*\right). \qquad (9)$$

This equation can be solved using standard least squares techniques yielding

$$\hat{\mathbf{u}} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{y}. \qquad (10)$$

Unfortunately, the least squares formulation can be extremely ill-conditioned[1]. In fact, for large oversampled data sets, the matrix $\mathbf{H}^T\mathbf{H}$ will usually have small eigenvalues, and the inverse problem will be ill-conditioned in general[1].

Consider instead the observation noise case

$$\mathbf{y} = \mathbf{H}\mathbf{u} + \mathbf{n} \qquad (11)$$

where $\mathbf{n}$ is independent identically distributed (IID) Gaussian noise. Further, assume that $\mathbf{u}$ is IID Gaussian, so that

$$E\{\mathbf{n}\mathbf{n}^T\} = \sigma^2 I \quad E\{\mathbf{u}\mathbf{u}^T\} = v^2 I \qquad (12)$$

In this case, it can be shown that the minimum variance(MV) estimator is[6] $\hat{\mathbf{u}} = E\{\mathbf{u}\mathbf{y}^T\}\left[E\{\mathbf{y}\mathbf{y}^T\}\right]^{-1}\mathbf{y}$ or

$$\hat{\mathbf{u}} = \mathbf{H}^T(\mathbf{H}\mathbf{H}^T + \gamma^{-2}I)^{-1}\mathbf{y} \qquad (13)$$

where $\gamma \equiv v^2/\sigma^2$ can be determined from the data[7]. This solution is also shown to be the minimum mean square error Bayes estimate under Gaussian priors, as well as the Ridge Regression estimate[1][2][3], so other statistical methods exist for estimation of γ^2 from the data. The estimator (13) is well-conditioned in general, and it has many other desirable properties[1].

Although the solution (13) requires $O(M^3n^3)$ operations in general, observe that by using the raised cosine pulse as a basis function, $\mathbf{H}$ becomes approximately banded with bandwidth $w \ll Mn$ such that for some $0 < \epsilon \ll 1$,

$$\left|p\left(t_i - t_j^*\right)\right|^2 < \epsilon \qquad \text{for all } |i-j| \ge w/2. \qquad (14)$$

If this condition is true, then we can use banded matrix solution techniques to approximate the true solution in $O(w^2Mn)$ operations. It can be shown that for fixed ϵ, this solution will be roughly $O(Mn)$ as $Mn \to \infty$, provided the banded approximation holds. For any $\alpha > 0.1$, the raised cosine pulse energy has this banded form with $w \ll Mn$ as $Mn \to \infty$, provided the largest gap between samples, $\max_{i,j}\left|t_i - t_j^*\right|$, is bounded. Even if this condition is violated, the matrix $\mathbf{H}$ remains near sparse and the computational bounds often hold.

5. The δ_{n-1} Method

Another important property of the raised cosine and sinc pulses is that they have zeros at kT for every $k \neq 0$, and $r(0) = 1$ for these pulses. If we assume $\delta_1 = 0$, and the noise is small, we see that

$$y(t_i) = y(t_i^*) \qquad \text{for } i = 1 + kn,\ k \ge 0.$$

Thus there are M values of $\mathbf{u}$ which are observed directly, resulting in a reduced computational requirement C_{n-1}

$$C_{n-1} = \frac{n-1}{n} C_n \qquad (15)$$

where C_r is the computational requirement for the δ_r method. For $n < 5$ this savings can be significant.

6. Edge-Effect Compensation

In general, these resampling estimators will exhibit more error at the ends of the interval. This is not surprising since the infinite series was simply truncated to form Mn equations. Alternately, it was implicitly assumed that $y(t) = 0$ for $t < t_1^*$ and $t > t_{Mn}^*$. The following modifications attempt to compensate for edge effects.

6.1. Case 1: Known outliers

For this case, assume that at least $w/2$ observed values $y(t)$ are known before and after the interval $[t_1, t_{Mn}]$ where w is the approximate matrix bandwidth as defined in (14). Form the augmented matrix equation

$$\mathbf{y} = \begin{bmatrix} H_L \\ \mathbf{H} \\ H_R \end{bmatrix} \mathbf{u} + \mathbf{n} = \mathbf{H}_+\mathbf{u} + \mathbf{n} \qquad (16)$$

where $\mathbf{H}_+$ is an $Mn + w$ by Mn rectangular band matrix with bandwidth w. This system can be solved efficiently using roughly $O(w^2 Mn)$ operations.

6.2. Case 2: Unknown outliers

For the case where there are only Mn observed values of $y(t)$, assume that the unknown values of $y(t)$ are elements of random vectors n_L, and n_R so that

$$\mathbf{y} = \mathbf{H}\mathbf{u} + E_L n_L + E_R n_R + \mathbf{n} \qquad (17)$$

where n_L and n_R are zero mean with

$$E\{n_L n_L^T\} = E\{n_R n_R^T\} = \sigma_y^2 I \qquad (18)$$

and $\mathbf{y}$, $\mathbf{H}$, E_L, and E_R are known. In addition assume that n_L, n_R, $\mathbf{n}$, and $\mathbf{u}$ are mutually independent. Therefore

$$E\{\mathbf{u}\mathbf{y}^T\} = E\{\mathbf{u}\mathbf{u}^T\mathbf{H}^T + \mathbf{u}n_L^T E_L^T + \mathbf{u}n_R^T E_R^T + \mathbf{u}\mathbf{n}^T\}$$

$$E\{\mathbf{y}\mathbf{y}^T\} = v^2\mathbf{H}\mathbf{H}^T + \sigma_y^2 E_L E_L^T + \sigma_y^2 E_R E_R^T + \sigma^2 I \quad (19)$$

Then the MV estimator is

$$\hat{\mathbf{u}} = v^2\mathbf{H}^T\left(v^2\mathbf{H}\mathbf{H}^T + \sigma_y^2 E_L E_L^T + \sigma_y^2 E_R E_R^T + \sigma^2 I\right)^{-1}\mathbf{y}.$$

Although the matrix (19) is not the same form as before, it is still approximately band form with band width at most $2w$. Therefore this system can be solved efficiently using roughly $O(w^2 Mn)$ operations.

6.3. Case 3: Truncation

For this case, assume that the constraint $y(t) = 0$ for $t < t_1^*$ and $t > t_{Mn}^*$ results in unknown edge transients which decay roughly exponentially from the edges of the interval. Perform a standard estimate without any outlier compensation, and then truncate the ends of the solution vector.

6.4. Case 4: Linearly Interpolated Edges

Since $1/t^3$ convergence of the $p(t)$ series is not fast enough to prevent edge transients, we can try replacing the heavily truncated edge series equations with a more quickly converging series. For the example below, the $p(t)$ series was replaced with a zero-th order interpolating spline, or, linear interpolation. Typically, the first few and last few rows of $\mathbf{H}$ are based on $p(t) = p'(t)$, where

$$p'(t) = 1 - |t/T| \qquad \text{for } -T \leq t \leq T \qquad (20)$$

and zero otherwise. While the $p'(t)$ series is suboptimal, it has the effect of reducing estimation error due the series truncation.

7. Examples

7.1. Resampling error versus δ and ω

The δ_{n-1} method was used, where $n = 2$, $T = 1$, and $M = 64$. Note there is only a single δ_i parameter, δ, and the observed samples $\mathbf{y}$ come in groups of n, or pairs. The goal is to estimate the resampled $\mathbf{u}$ where

$$\mathbf{u} = [y(1), y(2), \ldots, y(128)]^T.$$

The function $y(t)$ was chosen to be $y(t) = 0.9\cos(\omega t) + 0.3\sin(\omega t)$, where ω was varied to illustrate frequency domain characteristics. The truncation method was used to determine relative error, in dB, which is defined as

$$RE \equiv 10\log_{10}\left(\frac{\sum_i (\mathbf{u}_i - y(i))^2}{\sum_i (y(i))^2}\right). \qquad (21)$$

In the truncation method, only the middle portion of the solution is used. For this example, the summations in (21) cover only $19 \leq i \leq 109$. The raised cosine pulse $p(t)$ was used as a basis function with $\alpha = 0.4$. Due to the symmetry of $p(t)$ about zero, the performance of the algorithm is highly symmetrical about $\delta = 1$. Also, the error near $\delta = 1$ is vanishing $\rightarrow 0$, since the interpolation is very close to observed data. For $\alpha = 0.4$, the raised cosine unity passband is $\omega \leq (1-\alpha)\omega_0/2 = 1.88$, so higher ω results in distortion. The RE surface is shown in Figure 1 with d = δ, and w = ω. Figure 2 shows the same computation for the least squares estimator (10) above. Note that the region near d=2 is invalid due to ill-conditioning.

7.2. Maximum Absolute Error

Absolute error is also an important issue. The maximum absolute error was computed using the parameters above, with the exception that the ends were linearly interpolated (three samples at each block end) rather than truncated. The quantity plotted in Figure 3 is

$$MAE \equiv 10\log_{10}\left(\frac{\max_i |\mathbf{u}_i - y(i)|}{\max_i |y(i)|}\right) \qquad (22)$$

which is based on the L_∞ norm. Since the MV resampling estimator is a function of L_2 norms, it is unrealistic to expect it to perform well under the L_∞ criterion. The -13 dB contour corresponds to a maximum error of 5%, and this region roughly constrains $\omega < 0.4$. Thus, in general, the data must be oversampled at a greater rate to meet this criterion.

8. Conclusion

The fast MV estimator described here is both accurate and well-conditioned for sufficiently oversampled signals. Although series truncation results in errors near the start and end of the estimate, several modifications are presented to reduce this effect. If the observation noise is Gaussian, then both bandlimited signal estimation (in noise) and resampling can be completed optimally in one step.

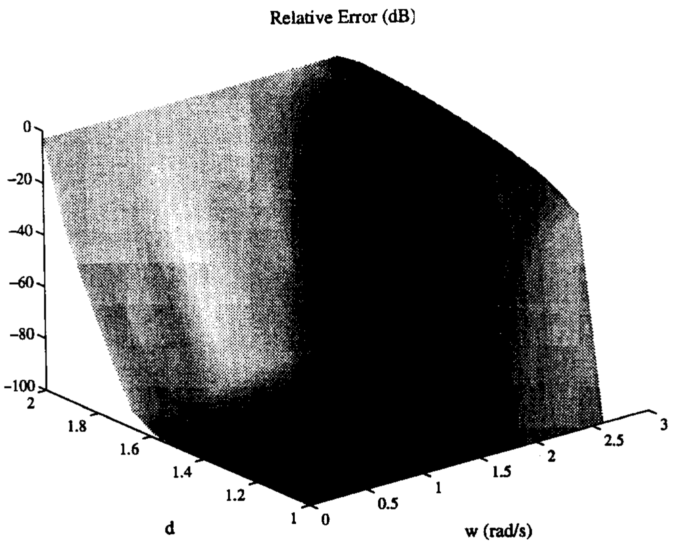

Figure 1: MV Error Surface ($\gamma = 10^{-5}$)

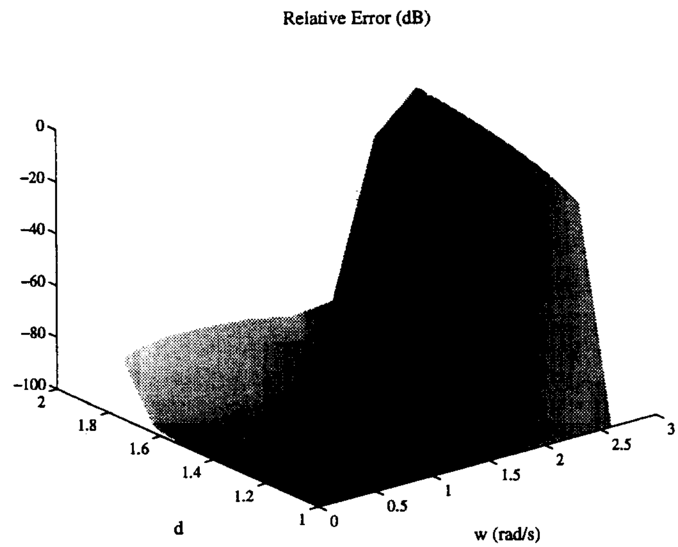

Figure 2: Least Squares Error Surface

9. References

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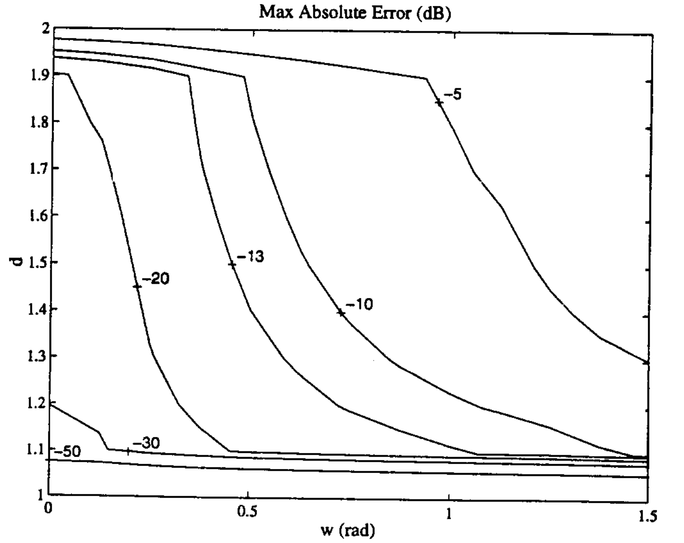

Figure 3: Maximum Error (Linearly Interpolated Edges)