RESOLUTION ANALYSIS IN SIGNAL RECOVERY

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ABSTRACT

Resolution analysis for the problem of signal recovery from finitely many linear samples is the subject of this paper. The classical Rayleigh limit serves only as a lower bound on resolution since it does not assume any recovery strategy and is based only on observed data. We show that details finer than the Rayleigh limit can be recovered by simple linear processing that incorporates prior information. We first define a measure of resolution based on allowable levels of error that is more appropriate for current signal recovery strategies than the Rayleigh definition. In the practical situation in which only finitely many noisy observations are available, we have to restrict the class of signals in order to make the resolution measure meaningful. We consider the set of bandlimited and essentially timelimited signals since it describes most signals encountered in practice. For this set we show how to precompute resolution limits from knowledge of measurement functionals, signal-to-noise ratio, passband, energy concentration regions, energy concentration factor, and a prescribed level of error tolerance. In the process we also derive an algorithm for high resolution signal recovery. We illustrate the results with an example.

1. INTRODUCTION

The problem of recovering signals from linear measurements arises in many applications, and several algorithms, linear and nonlinear, have been developed and analyzed for this problem. However, the fundamental question regarding the *resolution ability* of a recovery algorithm in the presence of noise and finitely many measurements is often left unanswered. Resolution ability is the ability to reproduce fine details such as, narrow peaks or closely spaced peaks in a signal. The study of resolution limits is important since, it could help us assess the effectiveness of a particular algorithm, and compare different algorithms in a rational manner. Moreover, understanding the relationship between resolution limits and the various components of a recovery problem and algorithm, could help us design better data acquisition schemes and algorithms.

The problem of resolution analysis is twofold: first, we require a meaningful measure of resolution ability; and second, we have to be able to analyze the performance of a reconstruction algorithm in terms of the defined resolution measure. The earliest definition of resolution limit is the *Rayleigh Resolution Limit,* which stipulates a resolution limit of δ if two equally strong point sources (impulse intensities), δ or more apart, are reproduced as peaks with at least a 19% intensity dip and sources less than δ apart are not reproduced as well [1]. This definition is based solely on the observed data and not on any recovery algorithm. It is an acceptable definition when there is no processing of the data to recover or enhance the features based on exploiting prior information. The Rayleigh limit is thus a lower bound on the resolution achievable. We might be able to do better with clever signal processing that exploits prior information, but we should always be able to achieve at least as much resolution as specified by the Rayleigh limit.

In this paper, we develop better estimates for resolution limits in signal recovery algorithms that take into account prior information, noise levels, and a finite number of measurements. We find that where infinitely many noise-free measurements are available, the resolution achievable is in fact independent of the width of the sampling pulse and depends only on the inter-sample distance. The Rayleigh limit, on the other hand, is dictated by the width of the sampling pulse. In the presence of noise however we find that the resolution limit depends on the method of regularization used in the recovery algorithm.

In the more practical situation in which only finitely many noisy observations are available the notion of exact recovery has to be abandoned and a new measure of resolution is necessary [2], [3]. We define a new measure based on allowable levels of worst-case error, and appeal to the Fourier uncertainty principle to bring out the relationship between resolution (detail) and bandwidth. In this sense our definition is similar in spirit to the classical Rayleigh resolution limit, but is based on a prescribed tolerance of the *relative error.* With a finite number of observations the worst-case error is unbounded and hence we have to restrict the search to a smaller set of signals.

We restrict the class of signals to the set of *bandlimited* and *essentially timelimited* signals, since it describes most signals encountered in practice. This set is characterized by the well known orthonormal family of functions called the *Prolate Spheroidal Wave Functions* and is known to be approximately finite dimensional, which enables us to seek reconstructions from a lower dimensional subspace of the space of bandlimited signals. Reduction in dimension causes an error in the reconstruction, which we call the *intrinsic error.* A second error is incurred while determining the parameters describing the lower dimensional reconstruction. The reconstruction error is then the sum of these two errors. We show that the worst-case values of these two errors can be pre-computed for each choice of reduced dimension. The error computation provides both an optimal choice of dimension and a precomputed bound on the resolution ability of the algorithm.

2. THE SIGNAL RECOVERY PROBLEM

We consider the problem of reconstructing 1-D continuous-index signals from discrete linear measurements. The results presented here can be easily generalized to multidimensional signals.

Let $L_2(R)$ be the space of finite-energy continuous-index signals. Let $\mathcal{B}_\delta$ be the subspace of all signals bandlimited to $P_\delta = [\frac{-\pi}{\delta}, \frac{\pi}{\delta}]$ and let B denote the orthogonal projection operator onto $\mathcal{B}_\delta$.

We address the problem of recovering a signal from $\mathcal{B}_\delta$ using discrete linear measurements. Let g_i be measurement signals giving measurements $y_d(i)$ as

$$y_d(i) = \langle x, g_i \rangle = \int_R x(t)\overline{g_i(t)}dt. \qquad (1)$$

Let T be the linear bounded operator on $\mathcal{B}_\delta$ representing the measurement process. Then

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$$Tx = y_d, \tag{2}$$

where y_d is the vector of measurements. If the number of measurements is finite, y_d lies in C^p, where p is the number of measurements, otherwise y_d lies in $l_2(Z)$, the space of finite-energy discrete-index signals.

In practice, the measurements are corrupted by noise. Let n_d denote the noise vector, then

$$z_d = y_d + n_d,$$

and the signal recovery problem is that of reconstructing x from z_d.

In the ideal case of infinitely many noise-free observations, we define resolution as follows:

Definition 2.1 (Resolution limit under ideal conditions) A reconstruction algorithm is said to have an *ideal resolution* of δ if signals bandlimited to $[-\frac{\pi}{\delta}, \frac{\pi}{\delta}]$ can be reconstructed perfectly under noise-free conditions. ■

By a simple frequency-domain analysis it can be shown that if Δ is the width of the sampling pulse and τ is the inter-pulse distance, the resolution limit, in the adequately-sampled case, is equal to τ instead of Δ as predicted by the Rayleigh limit [4]. In the noise-corrupted case however, the signal cannot (in general) be perfectly reconstructed and regularization is required. Regularization reduces resolution as seen in the example of Figure 1. In this example $\tau = \frac{1}{16}$ and $\Delta = \frac{1}{4}$ and, hence, a resolution limit of $\tau = \frac{1}{16}$ is achieved in the noise-free situation. However, with regularization to combat noise the signal cannot be recovered perfectly and hence resolution according to the above definition is affected. A new measure of resolution is needed

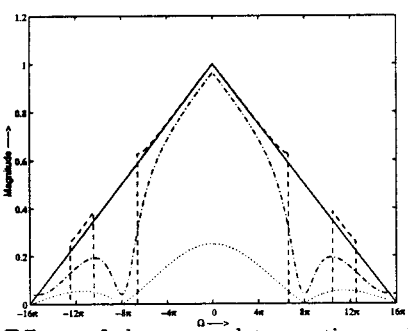

Figure 1: Effects of the spectral truncation and Tikhonov regularization in the frequency domain. —— true spectrum; - - - with spectral truncation; — · — with Tikhonov scheme; ··· sampling kernel spectrum.

that allows for imperfect reconstruction [2]. We develop a measure of resolution based on the maximum tolerable worst-case error as follows:

Definition 2.2 A reconstruction algorithm has *ϵ'-resolution* of δ or better if the worst-case normalized reconstruction error (over all $x \in \mathcal{B}_\delta$ and all n_d such that $\|n_d\| \leq \eta\|Tx\|$) is no larger than ϵ':

$$\frac{\|x - \hat{x}\|^2}{\|x\|^2} \leq \epsilon' \quad \forall x \in \mathcal{B}_\delta;\ \forall n_d : \|n_d\|^2 \leq \eta^2\|Tx\|^2.$$

Note that ϵ'-resolution is defined for a recovery algorithm and not for the recovery problem. In order to lower-bound the ϵ'-resolution limit for a particular recovery algorithm, we require tight upper bounds for the worst-case normalized reconstruction error.

3. RESOLUTION LIMIT IN THE PRACTICAL SITUATION

In practice, only finitely many measurements are available. Let p be the number of measurements. The measurement operator is $T : L_2(R) \rightarrow C^p$ and the reconstruction problem is formulated as follows :

Given T, δ, and measurement vector $z_d \in C^p$, find $x \in \mathcal{B}_\delta$ such that $Tx = z_d$.

The set of all signals in $\mathcal{B}_\delta$ satisfying $Tx = y_d$ is a linear variety, $\mathcal{V}$, with finite co-dimension p. The min-max optimal solution, which is also the minimum norm solution, is again given by

$$\hat{x}_{MNLS} = T^*(TT^*)^\dagger z_d \tag{3}$$

The operator TT^* is simply a $p \times p$ matrix.

Since the true signal can be any member of $\mathcal{V}$, the supremum of the normalized reconstruction error, $\frac{\|x-\hat{x}_{MN}\|}{\|x\|}$, is 1. Hence, we need to restrict the set of admissible signals appropriately to get a non-trivial bound on the worst-case error. However, having a finite number of measurements is often justified because most signals encountered in physical systems are *essentially timelimited.* Accordingly, we restrict our attention to such signals.

Let Γ be the concentration window and W be the windowing operator onto Γ. Let $G_{\epsilon,\delta}(\Gamma)$ denote the set of signals which are bandlimited to $[\frac{-\pi}{\delta}, \frac{\pi}{\delta}]$ and ϵ-essentially timelimited to Γ [5, 6]; i.e.,

$$G_{\epsilon,\delta}(\Gamma) \triangleq \{x \in \mathcal{B}_\delta : \|Wx\|^2 \geq (1-\epsilon)\|x\|^2\}. \tag{4}$$

The set $G_{\epsilon,\delta}(\Gamma)$ represents most signals encountered in physical imaging and information systems. Hence we have the following definition of resolution:

Definition 3.1 A reconstruction algorithm on concentration window Γ and concentration factor $1-\epsilon$ with $SNR \geq \frac{1}{\eta^2}$ will be said to have ϵ'-resolution of δ or better if $\forall x \in G_{\epsilon,\delta}(\Gamma)$ and $\forall n_d$ *s.t.* $\|n_d\| \leq \eta\|Tx\|$, $\frac{\|x-\hat{x}\|^2}{\|x\|^2} \leq \epsilon'$.

With $G_{\epsilon,\delta}(\Gamma)$ as the feasible set, the recovery problem becomes:

Given T, δ and the measurements $z_d \in C^p$, find $x \in G_{\epsilon,\delta}(\Gamma)$ such that $Tx = z_d$.

The set $G_{\epsilon,\delta}(\Gamma)$ has several interesting properties which can be exploited to determine resolution limits. Many of these properties are characterized by an orthonormal family of functions called the *Prolate Spheroidal Wave Functions* (PSWF), $\{\phi_i\}_{i=1}^{\infty}$, and the associated eigenvalues, $\{\lambda_i\}_{i=1}^{\infty}$ [5, 6]. Since the resolution limit of a recovery algorithm is based on the worst-case relative error, our objective is to find an algorithm that will minimize the worst-case relative error.

3.1. DIMENSIONALITY REDUCTION

It is well known that $G_{\epsilon,\delta}(\Gamma)$ is *essentially finite dimensional* and among all N-dimensional subspaces, the span of the first N PSWF, S_N^ϕ, optimally approximates $G_{\epsilon,\delta}(\Gamma)$ with an *intrinsic error*:[5, 6]

$$E(S_N^\phi) = \begin{cases} 1 & \lambda_{N+1} \geq 1-\epsilon > 0 \\ \frac{\lambda_1-(1-\epsilon)}{\lambda_1-\lambda_{N+1}} & \lambda_{N+1} < 1-\epsilon \leq \lambda_1 \end{cases}. \tag{5}$$

This suggests that we might be able to restrict our reconstructions to a finite-dimensional subspace of $\mathcal{B}_\delta$ (of dimension less than p) and still get low error reconstructions. Thus for a fixed dimension, r, the subspace spanned by

$\{\phi_i\}_{i=1}^{r}$ minimizes the worst-case relative error. Moreover the worst-case relative error $E(S_r^\phi)$ decreases with r. Thus it would seem that, given p observations, the best choice of a lower dimensional subspace to approximate $G_{\epsilon,\delta}(\Gamma)$ would be $S_p^\phi = \text{span}\{\phi_1, \cdots, \phi_p\}$. This choice would lead to p equations in p unknowns. Unfortunately, the p parameters required to describe the reconstruction from S_p^ϕ cannot be determined exactly from the observations z_d for two reasons. First, z_d are noise-corrupted in practice. Second, the observations z_d are linearly related to $x \in G_{\epsilon,\delta}(\Gamma)$ and not to the projection x_p of x onto S_p^ϕ. Thus, an additional error will be incurred in determining the parameters that describe the lower dimensional approximate. We next derive an expression for this error and its worst-case value $\Xi(r)$ for a fixed dimension r. We suggest choosing r to minimize $E(S_r^\phi) + \Xi(r)$.

3.2. WORST-CASE ERROR ANALYSIS

Consider the reconstruction based on an r-dimensional approximation of $G_{\epsilon,\delta}(\Gamma)$, where $r \leq p$, and let x_r be the projection of x onto S_r^ϕ; $x_r = \sum_{i=1}^{r} \alpha_i \phi_i$ and the approximation error $e_r \triangleq x - x_r = \sum_{i=r+1}^{\infty} \alpha_i \phi_i$. The measurements z_d are linearly related to x and are corrupted by noise n_d:

$$z_d = Tx_r + Te_r + n_d = A_r \alpha^r + (Te_r + n_d) \qquad (6)$$

where A_r is a $p \times r$ matrix with $A_r(i,j) = \langle \phi_j, g_i \rangle$, $\alpha^r = (\alpha_1, \cdots \alpha_r)^T$. The LS solution of $T\hat{x} = z_d$, $\hat{x} \in S_r^\phi$, is determined from the MNLS solution of $A_r \hat{\alpha}^r = z_d$, which is

$$\hat{\alpha}^r = A_r^\dagger z_d = \alpha^r + A_r^\dagger (Te_r + n_d). \qquad (7)$$

Thus the reconstruction is given by $\hat{x}_r = \sum_{i=1}^{r} \hat{\alpha}_i^r \phi_i$ and $\zeta_r = x_r - \hat{x}_r = \sum_{i=1}^{r} (\hat{\alpha}_i^r - \alpha_i)\phi_i$ is the additional error incurred in determining the α_i parameters. ζ_r has contributions from both Te_r and measurement noise n_d.

Thus, since $e_r \perp S_r^\phi$ and obviously $\zeta_r \in S_r^\phi$, the total reconstruction error is

$$\|x - \hat{x}_r\|^2 = \|x - x_r + x_r - \hat{x}_r\|^2 = \|e_r\|^2 + \|\zeta_r\|^2 \qquad (8)$$

It is clear that, $\forall x \in G_{\epsilon,\delta}(\Gamma)$, $\frac{\|e_r\|^2}{\|x\|^2} \leq E(S_\phi^r)$ given by (5).

Let $\rho_r^2 \triangleq \frac{\|TT^*\|}{\sigma_{min}^2(A_r)}$ and $b_r \triangleq \frac{\|A_r^\dagger Te_r\|^2}{\|x\|^2}$. Then,

$$\begin{aligned} \|\zeta_r\|^2 &\leq \|A_r^\dagger Te_r\|^2 + \|A_r^\dagger n_d\|^2 + 2\|A_r^\dagger Te_r\| \, \|A_r^\dagger n_d\| \\ &\leq (b_r + \rho_r^2 \eta^2 + 2\sqrt{b_r \rho_r^2 \eta^2})\|x\|^2 \\ &\triangleq \Xi(r)\|x\|^2. \end{aligned} \qquad (9)$$

Now all that remains to be done is to obtain a tight upper bound on $b_r \triangleq \frac{\|A_r^\dagger Te_r\|^2}{\|x\|^2}$ over $x \in G_{\epsilon,\delta}(\Gamma)$. Mathematically, our objective is to find:

$$\sup_{\substack{\sum_{i=1}^{\infty} \alpha_i^2 = 1 \\ \sum_{i=1}^{\infty} \alpha_i^2 (1-\lambda_i) \leq \epsilon}} \| \sum_{i=r+1}^{\infty} \alpha_i A_r^\dagger T \phi_i \|^2 \qquad (10)$$

This is a nonlinear infinite programming problem, and in general we can only seek approximate solutions. We first show in Theorem 3.1 that under mild assumptions the above problem can be approximated by a *finite variable* nonlinear programming problem. We then apply well known techniques to solve this problem.

Theorem 3.1 *Let $\gamma > 0$ and $\|x\| = 1$. There exists a positive integer N independent of the choice of $x \in G_{\epsilon,\delta}(\Gamma)$ such that*

$$\|A_r^\dagger Te_r\| - \| \sum_{i=r+1}^{N} \alpha_i A_r^\dagger T \phi_i \| \leq \gamma. \qquad (11)$$

Furthermore,

$$\sup_{x \in G_{\epsilon,\delta}(\Gamma)} \frac{\|A_r^\dagger Te_r\|}{\|x\|} = \gamma + \sup_{\substack{\sum_{i=1}^{N} \alpha_i^2 = 1 \\ \sum_{i=1}^{N} \alpha_i^2 (1-\lambda_i) = \epsilon}} \| \sum_{i=r+1}^{N} \alpha_i A_r^\dagger T \phi_i \| \qquad (12)$$

Theorem 3.1 serves to show that the solution to the infinite programming problem in (10) can be approximated closely by a sufficiently large finite variable problem. A popular method for solving this nonlinearly constrained quadratic program is the *Sequential Quadratic Programming* method, [7, 8]. Thus a bound on the worst-case normalized error, $\Theta(r)$, can be obtained by the sum of the intrinsic error and a bound on the worst-case $\|\zeta_r\|^2$, i.e.,

$$\Theta(r) = \min(1, E(S_\phi^r) + \Xi(r)). \qquad (13)$$

Thus, to ascertain whether a resolution of δ can be achieved given a set of p measurements we first compute the bound on the worst-case normalized error bounds for each dimension r ranging from 1 through p using the PSWF and the associated eigenvalues corresponding to $P_\delta = [\frac{-\pi}{\delta}, \frac{\pi}{\delta}]$ and determine the dimension r^* which gives the minimum error. If the worst-case error for this dimension is below the allowable error then we can claim that a resolution of δ can be achieved with the given set of measurements and noise level.

Remarks:

1. As a consequence of the above analysis we have a new algorithm for signal recovery based on dimension reduction guided by the bound on the worst-case reconstruction error.
2. The worst-case error-bound given by (13) does not depend on the data z_d. It depends only on the sampling functions, g_i, the bandwidth, $\frac{2\pi}{\delta}$, the sample spacing, τ, the noise level, η, and the choice of the dimension, r. Thus, the selection of the dimension and the determination of resolution can be made (off-line) before the measurements are taken.
3. Our analysis and definition of resolution are based on worst-case errors in a deterministic framework. Therefore, in general the reconstruction error can be expected to be lower than the predicted value.
4. The analysis holds true for all sampling patterns. Hence g_i, the i^{th} measurement function, does not have to be a shifted version of a single measurement function g_0. The only restriction is that the support of each g_i lies inside the concentration window, Γ.
5. We have assumed essential timelimitedness and strict bandlimitedness in our treatment, which can be easily changed to essential timelimitedness to Γ and essential bandlimitedness to P. The PSWF will still be the optimal sequences, [9] and all the results will still hold true, with minor modifications.
6. The PSWF have been studied in the classical setting of 1D signals with lowpass passband and contiguous concentration intervals. The three relevant properties of the PSWF, the dimensionality theorem, and thus, all our results, can be generalized to the more general setting of mD signals with P and Γ discontiguous [10].

Example 3.1 (1D setting) Consider reconstruction of 1D signals that are bandlimited to $[-4\pi, 4\pi]$ and have at least 99.5% of their energy concentrated in $[-2.0, 2.0]$, from 19 measurements taken with shifted unit rectangular pulses of width $\frac{1}{2}$ and inter-pulse distance $\frac{1}{4}$. Thus δ, Γ, and ϵ

are $\frac{1}{4}$, $[-2.0, 2.0]$ and 0.005 respectively in the definition of $G_{\epsilon,\delta}(\Gamma)$, while p and τ are 19 and $\frac{1}{4}$.

Note that the width of the sampling pulses is $\frac{1}{2} = 2\delta$. Thus the Rayleigh resolution limit is 2δ. Let the error tolerance be 10%. We will show that, depending on the SNR, a resolution limit of $\tau = \frac{1}{4}$ can be achieved using the algorithm based on the worst-case error analysis.

The terms $E(S_r^{\phi})$, b_r, ρ_r and $\Theta(r)$ are computed for each dimension r for which $E(S_r^{\phi}) < 1$ and tabulated below.

r	$E(S_r^{\phi})$	b_r	noise-factor ρ_r	$\Theta(r)$ SNR=40dB
13	0.2203	0.1905	4.3953	0.4511
14	0.0379	0.0430	5.4552	0.1065
15	0.0100	0.0189	7.0128	0.0531
16	0.0046	0.0338	9.4899	0.0823
17	0.0034	0.0788	13.2808	0.1744
18	0.0032	0.1103	17.6118	0.2615
19	0.0031	0.1124	27.8416	0.3797

From the table we observe that, with a 40dB SNR, the optimal dimension for this signal recovery problem is 15, and the worst-case normalized error is bounded above by 0.0531. In fact, with a 10% error allowance, a resolution of at least $\tau = 0.25$ can be achieved by the above algorithm whenever SNR is greater than 32dB. All these computations can be done off-line since they do not depend on the actual observed data.

We now test the reconstruction algorithm using the optimal dimension determined above on a specific signal:

$$x(t) = \left(\frac{sin(0.4\pi t)}{0.4\pi t}\right)^2 + 0.2\frac{sin(0.1\pi t)}{0.1\pi t}cos(3.5\pi t)$$

A plot of the signal is shown in Figure 2. The highest frequency in $x(t)$ is 3.6 radians. Notice that $x(t)$ is a low frequency signal with a low energy high frequency ripple. The frequencies are selected such that the low frequency component falls below the Rayleigh limit and hence is captured by the observations whereas the *high frequency ripple is much above the Rayleigh limit* and hence is not seen in the observations. A high resolution reconstruction should resolve the high frequency ripple also. $x(t)$ has 99.69 % of its energy inside the concentration interval $[-2.0, 2.0]$, i.e., $\epsilon = 0.0031$. Thus it belongs to the set $G_{\epsilon,\delta}(\Gamma)$ considered in this example. We take 19 observations in the concentration window with shifted versions of the sampling function. The observations are shown in Figure 2. Notice that the high frequency ripple is completely lost in the observations. The reconstruction, $\hat{x}$, is computed using the algorithm with an optimal dimension of 15 determined by the worst-case error analysis. The normalized error for this reconstruction, $(\frac{\|x-\hat{x}\|}{\|x\|})^2$, is computed to be 0.0068, which is much below the worst-case error bound of 0.0531.

4. CONCLUSIONS

A resolution analysis for signal recovery from finitely many discrete, noise-corrupted, linear measurements is presented. A new measure for resolution is introduced, which is more appropriate than the Rayleigh resolution limit for current signal recovery algorithms. This resolution measure is based on a prescribed tolerance of relative error in the reconstruction, and unlike previous definitions is able to bring out the extent to which time or spatial domain features can be recovered by an algorithm. The computation of resolution limits reduces to the computation of the worst-case relative error in the recovered signal. By suitably constraining the class of feasible signals, the worst-case error is expressed as the solution of a finite-variable nonlinear program. The analysis and example show that details finer than the Rayleigh resolution limit can be recovered by simple linear processing even in practical situations with finite, noise-corrupted data. In the process, we derive an algorithm for high resolution reconstruction (from linear observations) and show how one can precompute worst-case error bounds and the resolution limit for the algorithm.

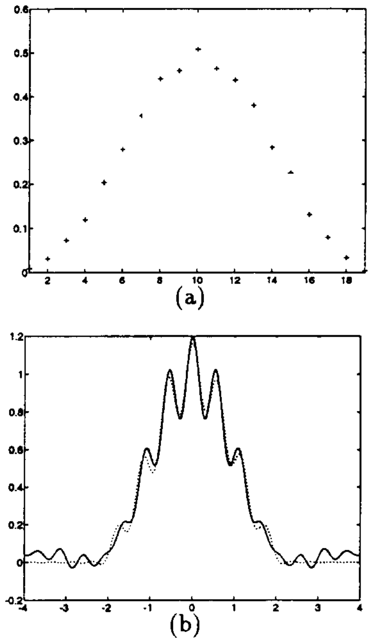

Figure 2: Reconstruction of the test signal using the optimal dimension determined by the worst-case error analysis. The SNR is 40db. (a) + Observations (b) —— signal, $x(t)$; ······ reconstruction, $\hat{x}(t)$.

5. REFERENCES

[1] P. A. Jansson, *Deconvolution: With Applications in Spectroscopy*, Academic Press, Orlando, Florida, 1984.

[2] L. S. Joyce and W. L. Root, "Precision bounds in superresolution processing", *J. Opt. Soc. Am. A*, vol. 1, no. 2, pp. 149–168, February 1984.

[3] W. L. Root, "Ill-posedness and precision in object-field reconstruction problems," *J. Opt. Soc. Am. A*, vol. 4, pp. 171–179, January 1987.

[4] S. Dharanipragada and K. S. Arun, "Resolution limits in signal recovery", *submitted to IEEE Trans Sig. Proc.*

[5] D. Slepian and H. O. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty - I", *Bell Syst. Tech. J.*, vol. 40, pp. 43–64, Jan. 1961.

[6] H. J. Landau and H. O. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty - II", *Bell Syst. Tech. J.*, vol. 40, pp. 65–84, Jan. 1961.

[7] D. G. Luenberger, *Optimization by Vector Space Methods*, Wiley, New York, 1969.

[8] P. E. Gill, W. Murray, and M. H. Wright, *Practical Optimization*, Academic Press, New York, 1981.

[9] A. A. Melkman and C. A. Micchelli, "Optimal estimation of linear operators in Hilbert spaces from inaccurate data," *SIAM J. Numer. Anal.*, vol. 16, pp. 87–105, February 1979.

[10] S. Dharanipragada, "Time-bandwidth dimension and its application to signal reconstruction," Master's thesis, University of Illinois, Urbana, IL, 1991.