ANALYSIS AND DESIGN OF MINIMAX-OPTIMAL INTERPOLATORS

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ABSTRACT

We study the problem of interpolating a bandlimited signal from its nonuniform samples. We consider a class of interpolation algorithms that includes the least-squares optimal interpolator proposed by J. L. Yen, and we derive a closed-form expression of the interpolation error for interpolators of this type. The expression for the interpolation error shows that the error depends on the eigenvalue distribution of a matrix, which is specified by the set of sampling points. We notice that the usual sinc-kernel interpolator is an approximation to the Yen interpolator, and we suggest a method of choosing the weighting coefficients in the sinc-kernel interpolator. The new sinc-kernel interpolator is superior to the sinc interpolator with the usual Jacobian (area) weighting and is far easier to implement than the Yen interpolator.

1. INTRODUCTION

The problem of signal reconstruction from nonuniformly sampled data can be found in various contexts, such as in the design of irregularly-spaced antenna arrays and the reconstruction of signals for cases with missing samples. Generalizing to two dimensions, interpolation from non-Cartesian data grids is an important problem arising in various Fourier imaging problems, such as tomography, synthetic aperture radar, and radio astronomy.

In most signal processing applications, the original signal to be reconstructed from nonuniform samples is modeled as bandlimited. Many types of interpolation algorithms have been devised for the reconstruction of bandlimited signals from nonuniform samples. Among the available algorithms, the sinc kernel figures prominently. A commonly used interpolation formula using the sinc kernel is

$$x_L(t) = \sum_{i=1}^{L} b_i x(t_i) sinc(\sigma(t - t_i)), \qquad (1)$$

where the b_i's are chosen to be the sample spacings (Jacobian) around the nonuniform sample locations t_i. But, this algorithm is not optimal in any usual sense. An optimal (in the least-squares sense) bandlimited interpolation algorithm was first derived by Yen [1]. The computation in the Yen interpolator involves the inversion of an $L \times L$ matrix where L is the number of samples. When L is large, the matrix often becomes highly ill-conditioned, so that regularization must be used to reduce the numerical errors in inversion. The regularization results in smoothing of the signal spectrum at high frequencies [2]. Although Yen interpolation is best in theory, there are extreme problems in computing the interpolated values numerically, which prevent realization of the expected performance. Hence, the Yen interpolator is of rather modest use in practice.

The Yen interpolation algorithm has been derived using several different approaches [3, 4, 5]. In [3], the algorithm was derived by assuming data matching at the sample points and finding the minimum norm solution. The same algorithm was also derived in this paper by optimization in the frequency domain, using a linear time-varying system model. In [4], the algorithm was derived by the so-called optimal recovery approach. Although it is not clear in [4], the linearity assumption was necessary to show that the unique, optimal algorithm is the pseudo inverse of the sampling operator. Otherwise, all linear varieties of the pseudo inverse can be optimal. In [5], the algorithm was derived by minimizing the least-squares error assuming a special form of the interpolation algorithm. The minimax optimality criterion used in [5] can be shown to be equivalent to that defined in [4]. However, it can be shown that the optimality criterion used in both of these papers is inadequate for measuring the performance of interpolation algorithms. In [6], the optimal recovery approach was used with the explicit assumption that the optimal algorithm is linear.

In this paper, we develop an approximate form of the Yen interpolator, which is easier to implement than the Yen interpolator, and less sensitive to noise. In our approach, we first give a measure of interpolator performance and then define the optimal interpolator to be that which minimizes the worst-case error. It is shown that the Yen interpolator is optimal when there is no restriction on the form of the interpolator. The work in this paper differs from the previous work in that we do not assume either data matching or linearity of the optimal interpolator for the derivation of the algorithm. Furthermore, we show that the optimality criterion stated in [4] and [5] is not useful, and we suggest a new optimality criterion. Our derivation of the optimal interpolator is based on this new definition of minimax optimality. Then, we seek an interpolator having a restricted form, which can be computed more easily and stably. The optimal choice of the parameters in the approximate interpolator is given by clustering the eigenvalues of a matrix product, around 1. We suggest a simple method that

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achieves this clustering, and we quantify the performance of this scheme through numerical simulation.

2. PROBLEM STATEMENT

Consider the problem of interpolation of a bandlimited signal x from its nonuniformly spaced samples, using an algorithm A. Let $S : X \rightarrow Y$ be the sampling operator, where X is the domain and Y is the range, respectively. In our discussion, X is the set of all signals bandlimited to $|\omega| < \sigma$, and $Y = \mathcal{R}^L$. We define the sampling operator $Sx = \{x(t_1), \cdots, x(t_L)\}$; t_i distinct. The interpolation error can be defined as

$$\|x - ASx\|, \tag{2}$$

where $\|\cdot\|$ is an appropriate norm. If X is a Hilbert space and $\Re(S)$, the range space of S, is closed in Y, X admits the orthogonal decomposition

$$X = \mathcal{N}(S) \oplus \mathcal{N}^{\perp}(S), \tag{3}$$

where $\mathcal{N}(S)$ is the null space of S and $\mathcal{N}^{\perp}(S)$ is its orthogonal complement. Any $x \in X$ can be decomposed as follows in a unique way: $x = x_N + x_S$, where $x_N \in \mathcal{N}(S)$ and $x_S \in \mathcal{N}^{\perp}(S)$.

In the previous literature [4, 5], the minimax optimal interpolator A_{opt} has been defined to be the one satisfying

$$\inf_{A \in \mathcal{A}} \sup_{x \in X_1} \|x - ASx\| = \sup_{x \in X_1} \|x - A_{opt}Sx\|, \tag{4}$$

where $\mathcal{A}$ is the class of algorithms under consideration, and X_1 is the subset of X having elements with unit energy. If the null space of S, $\mathcal{N}(S)$, is empty, then the sampling operator is invertible, so that

$$\inf_{A \in \mathcal{A}} \sup_{x \in X_1} \|x - ASx\| = 0. \tag{5}$$

If $\mathcal{N}(S) \neq \emptyset$, let $A(0) = z^A$ for each $A \in \mathcal{A}$. We can write $z^A = z_N^A + z_S^A$, where $z_N^A \in \mathcal{N}(S)$ and $z_S^A \in \mathcal{N}^{\perp}(S)$. For the signal $x = -z_N^A/\|z_N^A\| \in \mathcal{N}(S) \cap X_1$, $\|x - ASx\|^2 = 1 + \|z^A\|^2 \geq 1$. So, we have

$$\inf_{A \in \mathcal{A}} \sup_{x \in X_1} \|x - ASx\| \begin{cases} = 0 & \text{if } \mathcal{N}(S) = \emptyset \\ \geq 1 & \text{otherwise} \end{cases}. \tag{6}$$

Choosing $ASx = 0$ for all $x \in X_1$ would give $\inf_{A \in \mathcal{A}} \sup_{x \in X_1} \|x - ASx\| = 1$. Any useful algorithm should be better than this trivial algorithm. In the minimax sense, a reasonable algorithm should satisfy $\sup_{x \in X_1} \|x - ASx\| = 1$ if $\mathcal{N}(S) \neq \emptyset$ (in which case $z^A = 0$ or $A(0) = 0$.) This implies the optimality criterion defined by (4) is not useful for measuring the performance of interpolation algorithms, because the minimax error is completely specified by S and is independent of A for reasonable A.

In this section, we slightly modify the definition of minimax optimality, to provide a useful measure of the performance of interpolation algorithms. The optimal algorithm A_{opt} should satisfy

$$\inf_{A \in \mathcal{A}} \sup_{x_S \in \mathcal{N}^{\perp}(S) \cap X_1} \|x_S - ASx_S\| = \sup_{x_S \in \mathcal{N}^{\perp}(S) \cap X_1} \|x_S - A_{opt}Sx_S\|, \tag{7}$$

where, comparing with (4), we have restricted the domain of the supremum to $\mathcal{N}^{\perp}(S) \cap X_1$. The analysis below justifies this modification to the error criterion.

Since S is linear,

$$ASx = A(Sx_N + Sx_S) = ASx_S. \tag{8}$$

Write $ASx_S = y_N + y_S$ where $y_N \in \mathcal{N}(S)$ and $y_S \in \mathcal{N}^{\perp}(S)$. Then, we have

$$\begin{aligned} \|x - ASx\|^2 &= \|x_N + x_S - ASx_S\|^2 \\ &= \|x_N - y_N\|^2 + \|x_S - y_S\|^2. \end{aligned} \tag{9}$$

Since the interpolation error depends on the energy of the original signal, we restrict the signals to have a fixed energy E, without loss of generality. x_S can be exactly recovered from the available samples, since S is invertible when restricted to $\mathcal{N}^{\perp}(S)$. Let $\underline{x} = Sx$. Then x_S can be found using the minimum-norm inverse of S as

$$x_S = S^*(SS^*)^{-1}\underline{x}, \tag{10}$$

where S^* is the adjoint operator of S. It will be shown in the next section that the energy of a signal lying in $\mathcal{N}^{\perp}(S)$ can be expressed in terms of its samples, and the energy of x_S is given by

$$\|x_S\|^2 = \underline{x}^T \Phi^{-1} \underline{x}, \tag{11}$$

where Φ is the matrix whose $(i,j)^{th}$ element is $sinc(\sigma(t_i - t_j))$. Since x is assumed to have energy E, the energy of x_N is constrained to be

$$\|x_N\|^2 = E - \|x_S\|^2 = E - \underline{x}^T \Phi^{-1} \underline{x}. \tag{12}$$

Since we do not have any other information on x_N, x_N can be any signal in $\mathcal{N}(S)$ with the energy given by (12). To minimize the worst-case value of (9), we note that we must set $y_S = x_S$, and we choose y_N so that

$$\sup_{\|x_N\|^2 = E - \underline{x}^T \Phi^{-1} \underline{x}} \|x_N - y_N\| \tag{13}$$

is minimized. This is achieved by setting $y_N \equiv 0$, since x_N lies in a balanced set. This choice of y_S and y_N corresponds to the interpolator proposed by Yen [1], and later studied by others [3, 4, 5]. Given samples of $x(t)$ at $t_1, \cdots, t_L$, the interpolated signal is given by

$$y(t) = \sum_{m=1}^{L} \sum_{n=1}^{L} \gamma_{mn} x(t_n) sinc(\sigma(t - t_m)), \tag{14}$$

where γ_{mn} is the $(m,n)^{th}$ element of the inverse of the matrix Φ.

An interpolation algorithm can be thought of as a method of finding an estimate of x_S. So, the performance of an interpolation algorithm must be measured by its ability to estimate x_S in $\mathcal{N}^{\perp}(S)$. The definition of A_{opt} in (7) follows from this consideration. In (7) we assumed x_S has unit energy without loss of generality.

3. ANALYSIS OF INTERPOLATION ERROR

Given a bandlimited signal $x(t)$, consider an interpolator of the form

$$x_L(t) = \sum_{m=1}^{L}\sum_{n=1}^{L} b_{mn}x(t_n)sinc(\sigma(t-t_m)). \quad (15)$$

The $\{b_{mn}\}$ are parameters of the interpolator, to be chosen. We assume that the $\{b_{mn}\}$ are symmetric, because a symmetric choice includes the Yen interpolator, and the analysis of the interpolation error simplifies. The following analysis can be modified, however, for an asymmetric choice of $\{b_{mn}\}$. The choice of $\{b_{mn}\}$ corresponding to the Yen interpolator has been proven optimal in the sense of many optimality measures [4, 6]. But, as noted in previous literature, the exact evaluation of the optimal $\{b_{mn}\}$ suffers from numerical ill-conditioning, especially when the data are noisy and the number of sample points is large [2]. Thus, in practice, we are forced to use non-optimal values for the $\{b_{mn}\}$. In this section, we derive a formula for the interpolation error as a function of the $\{b_{mn}\}$ and show how to select these parameters in a suboptimal, but well-conditioned manner that can still produce small interpolation error.

We begin by finding an expression for the error between the original signal x and the reconstructed signal x_L. Define the inner product between two σ-bandlimited signals $f(t)$ and $g(t)$ as follows:

$$(f,g) = 2\sigma\int_{-\infty}^{\infty} f(t)g(t)dt. \quad (16)$$

The norm induced by this inner product is $\|f\| = \sqrt{(f,f)}$. With simple algebra, it can be shown that

$$\|x - x_L\|^2 = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} x(\frac{\pi}{\sigma}i)x(\frac{\pi}{\sigma}j)p_{ij}, \quad (17)$$

where

$$p_{ij} = 2\sigma\int_{-\infty}^{\infty}\phi_i(t)\phi_j(t)dt \quad (18)$$

and

$$\phi_k(t) = sinc(\sigma(t-\frac{\pi}{\sigma}k)) - \sum_{m=1}^{L}\sum_{n=1}^{L} b_{mn}sinc(\sigma(t_n-\frac{\pi}{\sigma}k))sinc(\sigma(t-t_m)). \quad (19)$$

The interpolation error depends on the signal to be interpolated and the weighting coefficients b_{mn}, $m = 1,\cdots,L$, $n = 1,\cdots,L$. Hence, when the signal to be interpolated is unknown, the interpolation error is also unknown. We shall measure the performance of the interpolator using the optimality criterion described in (7). An explicit expression for the interpolation error is obtained by expressing the p_{ij}'s as

$$p_{ij} = \delta_{ij} - 2\sum_{m=1}^{L}\sum_{n=1}^{L} b_{mn}sinc(\sigma(t_n-\frac{\pi}{\sigma}i))sinc(\sigma(\frac{\pi}{\sigma}j-t_m))$$
$$+ \sum_{m=1}^{L}\sum_{n=1}^{L}\sum_{p=1}^{L}\sum_{q=1}^{L} b_{mn}b_{pq}sinc(\sigma(t_n-\frac{\pi}{\sigma}i)) \cdot sinc(\sigma(t_q-\frac{\pi}{\sigma}j))sinc(\sigma(t_m-t_p)). \quad (20)$$

Substituting (20) into (17), and letting $\underline{x} = [x(t_1)\cdots x(t_L)]^T$, $B = [b_{ij}]$, and $\Phi = [sinc(\sigma(t_i-t_j))]$, gives an expression for the interpolation error in matrix notation:

$$\|x - x_L\|^2 = 1 - \underline{x}^T(2B - B\Phi B)\underline{x}. \quad (21)$$

We now proceed to find $x \in \mathcal{N}^{\perp}(S)\cap X_1$ giving the worst-case interpolation error. For $x \in \mathcal{N}^{\perp}(S)$, the energy of x can be written as

$$\|x\|^2 = \|\sum_{m=1}^{L}\sum_{n=1}^{L} x(t_n)\gamma_{mn}sinc(\sigma(t-t_m))\|^2 = \underline{x}^T\Phi^{-1}\underline{x}. \quad (22)$$

Since $x \in \mathcal{N}^{\perp}\cap X_1$ has unit energy, we have

$$\underline{x}^T\Phi^{-1}\underline{x} = 1. \quad (23)$$

Hence, the worst-case error occurs at the solution of the following constrained maximization problem:

$$\text{maximize} \quad 1 - \underline{x}^T(2B - B\Phi B)\underline{x},$$
$$\text{subject to} \quad \underline{x}^T\Phi^{-1}\underline{x} = 1.$$

This constrained optimization problem can be solved using the Lagrange multiplier method. At the solution of the problem, there exists a λ such that

$$-(B\Phi B - 2B)\underline{x} + \lambda\Phi^{-1}\underline{x} = 0. \quad (24)$$

Multiplying by Φ and rearranging terms, we obtain

$$(2\Phi B - (\Phi B)^2 - \lambda I)\underline{x} = 0. \quad (25)$$

This implies λ is an eigenvalue of the matrix $2\Phi B - (\Phi B)^2$, and x is the eigenvector corresponding to that eigenvalue. Thus, the interpolation error can be written as:

$$\|x - x_L\|^2 = 1 - \underline{x}^T(2B - B\Phi B)\underline{x} = 1 - \lambda. \quad (26)$$

The worst-case error occurs when x is the eigenvector corresponding to the minimum eigenvalue of $2\Phi B - (\Phi B)^2$, giving the worst-case error $\sqrt{1-\lambda_{min}}$. To minimize the worst case error, we must choose B so that λ_{min} is maximized. This is achieved by clustering the eigenvalues of ΦB around 1. The best case is when $B = \Phi^{-1}$, which corresponds to the Yen interpolator.

4. A NEW INTERPOLATOR

The interpolator in (1) falls within the category of interpolators considered above, by setting $b_{mn} = 0$, $m \neq n$ in (15). This choice of b_{mn} corresponds to using a diagonal matrix for B. This choice for B seems to be intuitively reasonable since the interpolated value at a particular point should depend heavily on samples near to that point, and less on samples far from it.

In view of the optimality criterion derived above, the optimal B for an interpolator of this form is a diagonal matrix which minimizes $\|\Phi B - I\|_S$, where $\|\cdot\|_S$ is the spectral norm of the matrix. Since minimization of the spectral norm is not easy, we explore using the Frobenius norm instead, to measure the closeness of ΦB to I. Although minimization of the Frobenius norm does not guarantee the minimum value of the spectral norm, it will provide an approximate solution to the problem. The minimum of $\|\Phi B - I\|_2$ is achieved when B is of the form

$$\begin{aligned} B &= diag(b_1, \cdots, b_L) \\ b_i &= \left[\sum_{j=1}^{L} sinc^2(\sigma(t_j - t_i)) \right]^{-1}. \end{aligned} \quad (27)$$

We conducted a simulation to demonstrate the performance of this new, suboptimal interpolator. We compared the result with the sinc-kernel interpolator with Jacobian weighting. For this simulation, a set of $L = 16$ samples was used with average sampling interval $T = 1$ s, giving a nominal bandwidth $\sigma = \pi$ rad/sec. For sample point distributions that were nearly uniform, we observed that the performance of the newly designed interpolator was similar to the interpolator with Jacobian weighting. Figure 1 shows the worst-case error for 100 randomly generated typical nonuniform sample point sets. The solid line represents the worst case error of the Jacobian-weighted interpolator and the dotted line represents the error for the newly designed interpolator. We notice that the interpolator proposed in this paper renders smaller worst-case error than the usual Jacobian-weighted interpolator. We notice that, in most cases, the Jacobian-weighted interpolator has worst-case error larger than the signal energy.

A second simulation compared the performances of the Yen interpolator without regularization (Yen-1), Yen with regularization ($\epsilon = 10^{-5}$, Yen-2), sinc interpolator with Jacobian weighting (Sinc-1), and sinc interpolator with weightings given by (27) (Sinc-2). The interpolator input was taken to be a sampled version of the superposition of 50 sinc functions having cutoff frequency $\sigma = \pi$ rad/s, having random amplitudes on the range [0,1], and randomly centered on the interval $[LT, 2LT]$. $L = 16$ was used in the simulation. Gaussian white noise was added to the signal samples. For each interpolator, a uniformly-sampled version of the output signal with sampling interval $T/16$ was reconstructed as an approximation of the continuous-time signal. To evaluate the reconstruction quality of the four interpolators, the signal-to-error (S/E) ratio was computed in dB. The S/E ratio was computed as the energy of the signal divided by the energy of the reconstruction error. The means and standard deviations of the S/E ratio, averaged over 100 trials (with different sampling instants and different signals), are reported in Table 1.

It is seen that the performance of the unregularized Yen interpolator drops abruptly as the noise level increases. For the regularized Yen interpolator, the performance is inferior to the unregularized one when there is no noise, but superior when there is noise. The two sinc-kernel interpolators are even less sensitive to noise. And, compared to the usual Jacobian weighted sinc-kernel interpolator, the interpolator proposed in this paper performed better for each SNR.

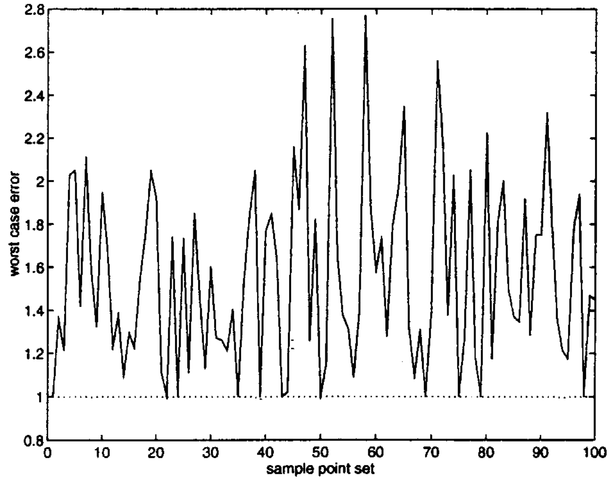

Figure 1: Worst-case interpolation error

Interpolation Method		SNR of data samples			
		no noise	40 dB	30 dB	20 dB
Yen-1	Mean	10.91	-27.95	-40.41	-42.90
	S. D.	4.78	22.95	23.06	23.01
Yen-2	Mean	9.99	-8.24	-17.37	-25.52
	S. D.	4.69	7.15	7.14	7.82
Sinc-1	Mean	2.22	2.30	0.22	-5.12
	S. D.	2.00	2.41	2.12	2.34
Sinc-2	Mean	4.51	4.35	3.40	-1.31
	S. D.	2.11	2.45	1.80	1.48

Table 1: Mean and standard deviation of S/E for different interpolators

5. References

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