AN APPROXIMATE ANALYSIS OF SIGMA-DELTA MODULATION OF A GAUSS-MARKOV PROCESS

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ABSTRACT

We address the problem of approximating the quantization noise spectra when a Gauss-Markov process is input to a sigma-delta modulator. The process is modeled using a state space approach. Fine quantization approximations are used to derive expressions for the output spectrum. Results similar to those of Gray's [1] analysis are obtained.

1. INTRODUCTION

Sigma-delta modulation has long been a preferred modulation scheme for oversampled analog to digital converters. The advantages of the system lie in the inherent tradeoffs between the resolution of the quantizers and the oversampling rate, robustness under input variations, and better spectral characterstics than those of delta modulation systems. In this paper we present an approximate, but simple, analysis of a sigma-delta modulation system which covers a large class of random and deterministic inputs. Our analysis is very general and gives results, despite being a simple and an approximate analysis, which are consistent with previous more rigorous analyses [1, 2, 3, 4, 5].

In [1, 2] Gray et al. have done an extensive analysis of the quantization noise spectra for the same inputs. In these analyses the quantizers are assumed to be uniform and it is assumed that the quantizers do not overload. By restricting the input to the no-overload region the quantizer error is periodic and it is expanded as a Fourier series. After intensive analysis, involving Bessel functions, Gray [1] shows that the quantization noise spectra is not white. For a DC input the spectrum is purely discrete, with the locations and amplitudes of the spectrum heavily dependent on the input signal. For a sinusoidal input, the output spectra is not continuous or white. In [6] Candy and Benjamin had reached a similar conclusion for DC inputs by using a slightly simpler analysis.

Both Gray et al. and Candy et al. restrict their analysis to either a DC or a sinusoidal input. With the exceptions of Chou and Gray [3], Wong and Gray [4] and, more recently, Galton [5], almost all the analyses of sigma-delta modulation systems are limited to deterministic inputs. In [4] Wong and Gray considered the input to the single loop sigma-delta modulator to be an independent and identically distributed (i.i.d) Gaussian process which is unbounded in magnitude. By using a continuous time model and doing a rigorous analysis they found a closed form expression for the quantization noise spectra when the input is a constant signal overriden by a Gaussian noise. It was shown that the quantization noise spectra is smeared into bands in contrast to the discrete line spectrum in the DC case. Recently, Galton [5] has given a rigorous analysis for the case when the input to the sigma-delta modulator is a sequence consisting of a desired input plus an additive independent and identically distributed random component. Explicit expressions for the autocorrelation of quantization noise are also derived.

In this paper we develop an approximate theory to analyze the quantization noise spectra of a sigma-delta modulator when the input is a Gauss-Markov process of any arbitrary order. The approximations that we use are meant for an optimum quantizer with large number of levels and in that sense are different from the analysis of Gray et al. and Galton. An advantage of our approach is that the input is a Gauss-Markov process with any arbitrary rational spectra. Furthermore, in the limit as the bandwidth of a narrowband process goes to zero, we approach the sinusoidal input considered in [1, 2]. Finally, following the paradigm of the analysis of [7] and [8], we compare the time averaged smoothed error for sigma-delta modulation with that of other previously analyzed source coding schemes. We show that the performance of the sigma-delta modulation system is worse in terms of the smoothed error than those of the state quantization schemes and the differential quantization of state.

2. SIGMA DELTA MODULATION

We consider the case when a sampled Gauss-Markov process x_n is input to a sigma-delta modulator. For

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an n-th order Gauss-Markov process, the discrete time state equation is given by

$$\mathbf{S}_{n+1} \quad = \quad \alpha \mathbf{S}_n + \Gamma \mathbf{W}_{n+1}, \tag{1}$$

where $\mathbf{W}_{n+1}$ is a vector of independent, zero mean, unit variance Gaussian random variables. $x_n = C\mathbf{S}_{n+1}$, where $\mathbf{S}_{n+1}$ is the sampled state vector of the Gauss-Markov process [7]. Here $C = [1 \;\; 0 \;\; \cdots \;\; 0]^T$, Γ is a transformation matrix, and $\alpha = e^{A\tau}$, where A is the state space matrix and τ is the sampling interval.

Figure 1 gives the block diagram of a sigma-delta modulation system. The sigma-delta modulation loop gives the following equation :

$$u_n \quad = \quad C\mathbf{S}_{n-1} + (u_{n-1} - q(u_{n-1})), \tag{2}$$

where u_n is the state of the sigma-delta modulator. Finally, the decoder is given by $\hat{x}_n = q(u_n)$.

We are eventually interested in finding the quantization noise spectra. For this, we first find the autocorrelation of the quantization error. This is given by

$$\begin{aligned} R_u[l] &= E(u_n - q(u_n))(u_{n-l} - q(u_{n-l})) \\ &= E(u_{n+1} - C\mathbf{S}_n)(u_{n-l+1} - C\mathbf{S}_{n-l+1})^T, \end{aligned} \tag{3}$$

where we have used equation (2) in getting (3). Here $(u_n - q(u_n))$ is the quantization of the state of the sigma-delta modulator at the n-th time instance. The quantization noise spectra is given by the taking the Fourier transform of this autocorrelation $R_u[l]$, and is given by: $S_e(f) = \sum_{-\infty}^{\infty} R_u[l]e^{-2j\pi fl}$.

We use the same notation as in Gray's analysis [1]. If we consider a narrowband Gauss-Markov process; in the limit as the bandwidth of the narrowband process goes to zero, the narrowband process approaches a sinusoidal input with a constant amplitude and a statistically independent phase. Gray's analysis is restricted to DC or sinusoidal inputs as opposed to our analysis which holds good for any Gauss-Markov process with arbitrary rational spectrum. Our analysis is different from that of Gray's. We use optimum quantizers as opposed to the uniform quantizers used by Gray. Although the optimum quantizers can be viewed as a practical drawback; they also enable us to lift the restriction on the no overload. Furthermore, our analysis can also be extended to uniform quantizers.

We use the difference equation for the sigma-delta modulator (2) and the input state equation (1) to obtain recursive equations for cross expectations between $\mathbf{S}_n$ and u_{n-l}. We use the asymptotic quantization approximation that for two random variable X and Y, $E\{Y(X - q(X))\} \approx \frac{K_x}{N_x^2} E\{XY\}$, where N_x is the number of levels in the quantizer and K_x is a constant depending on the probability density of the random variable X; for example $K_x = 2.71$ when X is a zero mean, unit variance Gaussian random process.

After solving the difference equations recursively and under the assumption that the number of levels in the quantizer N_u is large, it can be shown [9] that the autocorrelation of the quantization noise in the sigma-delta modulation can be approximated by :

$$R_u[l] \approx \begin{cases} \frac{K_u^2}{N_u^4} C\alpha^l R_{ss}(0)C^T & l > 0 \\ \frac{K_u^2}{N_u^4} CR_{ss}(0)\alpha^{-T^l}C^T & l < 0. \end{cases} \tag{4}$$

The quantization noise spectra $S_e(f)$ is given by :

$$\begin{aligned} S_e(f) &= \sum_{l=0}^{\infty} R_u[l]e^{-j2\pi fl} + \sum_{l=0}^{-\infty} R_u[l]e^{-j2\pi fl} - R_u[0] \\ &\approx \frac{K_u^2}{N_u^4} CH(f)H(f)^* C^T, \end{aligned} \tag{5}$$

where $H(f) = (I - \alpha e^{-j2\pi f})^{-1}\Delta$, $\Delta\Delta^* = [R_{ss}(0) - \alpha R_{ss}(0)\alpha^T]$, $R_{ss}(0)$ is the covariance matrix of the state $\mathbf{S}_n$, and $\alpha = e^{A\tau}$. Equation (5) is a general expression for the quantization noise spectra when a Gauss-Markov process characterized by the state space matrices A, B, and C is input to a sigma-delta modulator.

2.1. A Second Order Narrowband Example

After having derived a general expression for the quantization noise spectra for a Gauss-Markov input, we consider, as an example, a second order narrowband process. The second order narrowband process is given by $A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}$ and $B = k\begin{bmatrix} 1 \\ \omega_n \end{bmatrix}$. For a fixed ω_n, as ζ decreases the bandwidth of the spectrum becomes narrower.

We present the case when $\omega_n = 0.4542$, $\zeta = 10^{-9}$ and $\tau = 1.0$. The frequency $\omega_n = 0.4542$ is the same as that of the sinusoidal input considered as an example by Gray [1]. Under our set of assumptions the quantization noise spectra follows the input spectra. The difference between the two spectra is more or less constant or white. Figure 2 shows the different spectra. This is the case when the input is a narrowband Gauss-Markov process with unit variance. Similar observation was made by Wong and Gray [4] for independent and identically distributed (i.i.d) Gaussian inputs. They showed that as the input variance of the i.i.d Gaussian sequence increases, the quantization noise spectra has primarily just one peak and the presence of harmonics diminshes.

Figure 3 shows the results of simulations for autocorrelation of the quantization noise. The simulations are done with a uniform quantizer whereas the theoretical results correspond to our analysis, which is based on optimum quantizers. As the dynamic range of the uniform quantizer increases, the autocorrelation decreases. But again, for a fixed number of levels, if the dynamic range becomes too large, the autocorrelation of quantizer shows a sinusoidal-like behavior. This case corresponds to Gray's [1] result of quantization noise spectra for a sinusoidal inputs with a binary quantizer in the sigma-delta modulator. Gray observes several lines in the spectra for a binary quantizer. These correspond to a mixture of sinusoids in the autocorrelation. In our simulations, if we let the dynamic range of our uniform quantizer become large, the quantizer behaves practically as a binary quantizer. Thus our autocorrelation does approach that obtained by Gray for a sinusoidal input to a sigma-delta modulator with a binary uniform quantizer. We should note, however, that our input is not exactly a sinusoid but a very narrowband Gauss-Markov process with peak at the same frequency as the sinusoid considered by Gray.

3. SIGMA-DELTA MODULATION PERFORMANCE IN TERMS OF SMOOTHED ERROR

In this section we compare the performance of the sigma-delta modulation system with some other source coding systems. In [7, 8] we analyzed the performance of many source coding systems by tracking and quantizing the state vector $\mathbf{S}(t)$. The state vector is sampled and its quantized version is used to find an estimate of the state. For a fixed digital channel transmission rate R, we consider the problem of minimizing the overall mean square error averaged over one sampling interval.

By doing our analysis in the paradigm of [7, 8, 9], we can compare the performance of this sigma-delta modulation system with other source coding schemes. We have analyzed the performance of several quantization schemes for a narrowband input random process. Figure 4 gives the relative performance of these schemes with respect to the sigma-delta modulation system for the second order example considered in the previous section. The smoothed error for sigma-delta modulation is the largest among all the schemes.

4. CONCLUSION AND DISCUSSION

In this paper we have analyzed a sigma-delta modulation system, with a Gauss-Markov input, using fine quantization techniques. This analysis is different from most analyses in literature since it assumes optimum as opposed to uniform quantizers used traditionally. Although difficult in implementation, the use of optimum quantizer assumption removes the requirement that the input lie in the no overload region. Furthermore, the approximations that we derive are very general and hold good for any Gauss-Markov input having any arbitrary rational spectrum; unlike many of the previous analyses [1, 6] which are restricted to either DC or a sinusoidal inputs. Finally, we compare the performance of the sigma-delta modulation system with other envelope quantization schemes with respect to the smoothed error.

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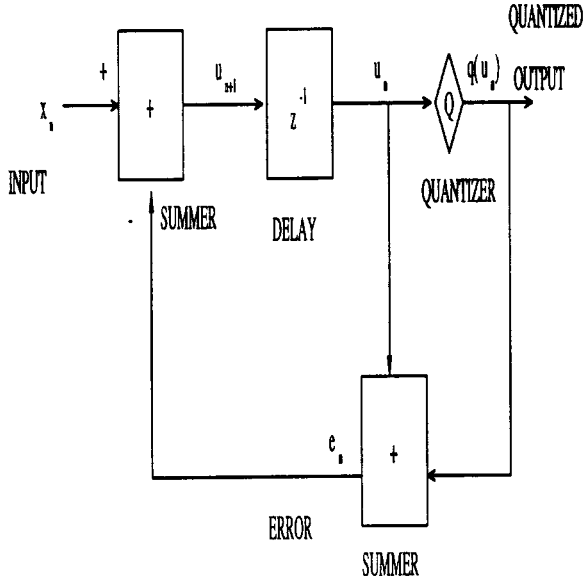

Figure 1: A Sigma-Delta Modulation System.

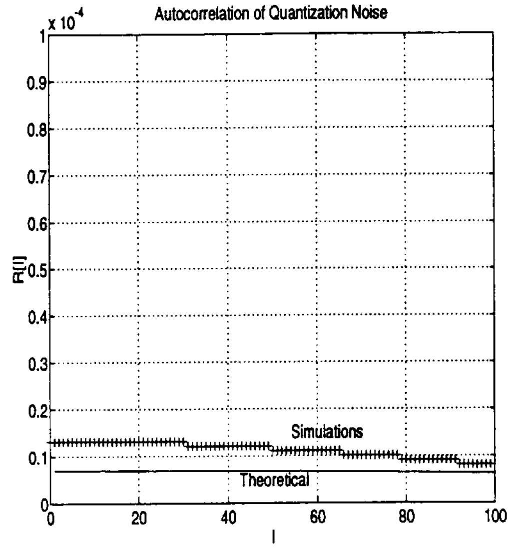

Figure 3: Autocorrelation of the quantization noise in sigma-delta modulator. The figure shows the simulations versus the theoretical results for the case when the input to sigma-delta modulator is a narrowband Gauss-Markov process with $\omega_n = 0.4542$, $\zeta = 10^{-9}$ and at sampling interval $\tau = 0.01$. The number of levels in the quantizer are 32.

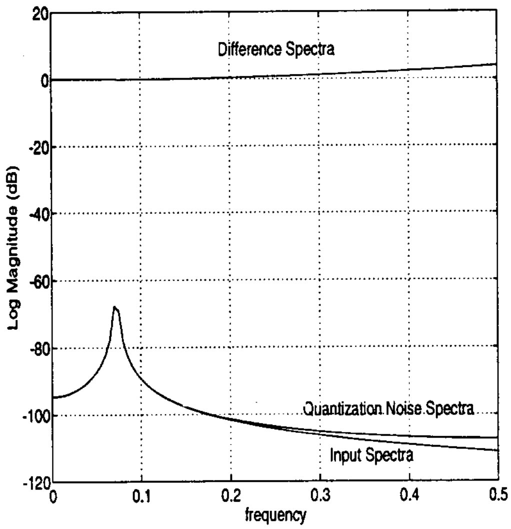

Figure 2: Quantization Noise Spectra for narrowband input at $\omega_n = 0.4542$, $\zeta = 10^{-9}$ and at $\tau = 1.0$. The quantization noise spectra follows the input spectra.

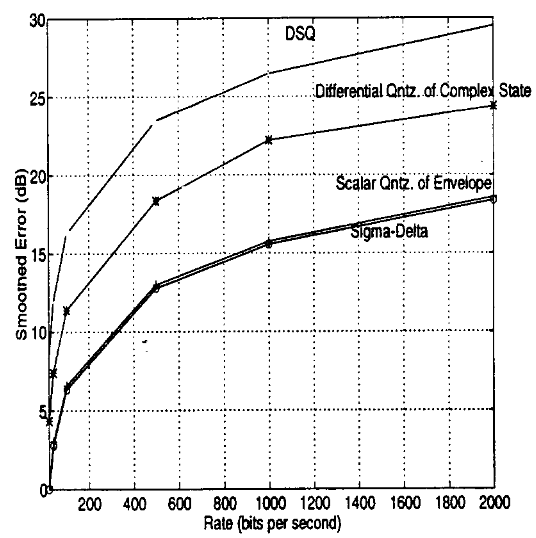

Figure 4: A comparison of performance of Sigma-Delta modulation system with scalar quantization of the envelope for the second order narrowband process considered with $\zeta = 0.001$ and $\omega_n = 10$. The figure plots $10 \log_{10} \frac{\zeta_{sm_o}}{\zeta_{sm}}$, where ζ_{sm_o} is the smoothed error for Sigma-Delta modulation at 20 bits per second. The four curves are (from top to bottom) : i) Differential State Quantization, ii) Differential Quantization of the Complex State, iii) Scalar Quantization of the complex envelope, and iv) Sigma-Delta Modulation.