ANALYTICAL DESIGN OF ZERO-PHASE CIRCULAR 2D FIR FILTERS

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ABSTRACT

This paper proposes a simple and efficient analytical design method for 2D non-recursive filters with circularly-symmetric and zero-phase frequency response. The design is achieved in the frequency domain and is based on prototype filters of two types: maximally-flat and Gaussian-shaped. The 2D FIR filter transfer function results directly in factorized form. Two types of 2D circular filters will be approached, namely low-pass with a specified bandwidth, flat top and steep transition region, and also narrow band-pass filters with specified peak frequency. Simulation results for the filtering of a biomedical image are also provided, showing the usefulness of these filters in image processing.

Index Terms— 2D filter design, circular FIR filters, approximation, frequency transformations

1. INTRODUCTION

The field of two-dimensional filters has largely developed during the last three decades and various design methods were proposed by researchers [1]. Generally the currently-used design methods for 2D recursive filters rely on 1D filter prototypes, using spectral transformations from *s* to *z* plane via bilinear or Euler approximations, with the aim to obtain a 2D filter with a desired frequency response [2]-[4]. A convenient and largely used tool for 2D filter design is also the McClellan transform [5], [6]. Filters with circular symmetry are very useful in image processing and various design methods for this particular class of 2D filters, both in FIR and IIR version, were developed in early papers like [7]-[12]. New advances regarding design and implementation were made in some recent works [13], [14]. Filters with elliptical symmetry [15] are anisotropic or directional filters and are derived as a more general case of the circular filters. They have important applications as well, for instance in remote sensing etc. [16]. A class of Gaussian-shaped circular filters implemented in cellular neural networks was approached in [17]. Maximally-flat circular IIR filters with adjustable bandwidth were proposed in [18].

2. ZERO-PHASE 1D FILTERS OBTAINED FROM ANALOG PROTOTYPES

Throughout this work only non-recursive (FIR) filters will be approached. The design will be based on a 1D prototype filter of a given type (low-pass, band-pass etc.), with specified parameters (cut-off or central frequency, selectivity, steepness etc.). Starting from an analog or digital filter with given specifications and derived from one of the currently-used approximations (maximally-flat or equiripple), the first design step is to obtain a rational approximation of its magnitude characteristics. Therefore the resulted prototype will be *zero-phase* throughout the frequency domain and its spatial frequency response will be a real-valued function. This is a desirable property and zero-phase filters are used in image processing because they introduce no phase distortions in the filtered image.

An analog filter of order *N* is described by the general transfer function in variable *s*:

$$H_P(s) = \frac{P(s)}{Q(s)} = \sum_{i=0}^{M} p_i \cdot s^i \Bigg/ \sum_{j=0}^{N} q_j \cdot s^j \qquad (1)$$

The proposed method is based on an analog prototype filter of a certain approximation type, with a transfer function of the general form (1).

In order to eliminate the phase variation of the filter prototype, we consider the magnitude characteristics, defined by the absolute value $|H_P(j\omega)| = |P(j\omega)| / |Q(j\omega)|$.

As a prototype, we can consider for instance a *Nth*-order Butterworth low-pass filter, which has the magnitude of its transfer function $H_B(\omega) = 1 \Big/ \sqrt{1 + (\omega/\omega_0)^{2N}}$; ω_0 is the cutoff frequency.

Now we look for a series expansion of the magnitude $|H_P(j\omega)|$, which has to be an approximation as accurate as possible on the frequency range $[-\pi, \pi]$. The most convenient for our purpose is the Chebyshev series expansion, because it yields an efficient approximation of a given function, which is uniform along the desired interval. The Chebyshev series of a given function on a specified interval can be easily found using a symbolic computation software like MAPLE. However, we will finally need rather a trigonometric expansion of $|H_P(j\omega)|$, namely in $\cos(n\omega)$, rather than a polynomial expansion in powers of the frequency variable ω. Therefore, prior to Chebyshev series calculation, the following variable change will be used:

$$\omega = \arccos(x/\pi) \Leftrightarrow x = \pi\cos(\omega) \qquad (2)$$

and so we get first the polynomial expansion in variable x:

$$|H_P(\arccos(x))| \cong \sum_{n=0}^{N} c_n \cdot x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_N x^N \qquad (3)$$

where the number of terms N is chosen large enough to ensure the desired precision (specified by the maximum error). The next step is to substitute back $x = \pi\cos(\omega)$ in the polynomial expression (3), therefore we obtain:

$$|H_P(\omega)| \cong H(\omega) = \sum_{n=0}^{N} b_n \cdot \cos^n(\omega) = b_0 + b_1\cos\omega + b_2\cos^2\omega + b_3\cos^3\omega + \ldots + b_N\cos^N\omega \qquad (4)$$

with $b_0 = c_0$ and $b_k = c_k \cdot \pi^k$. According to the fundamental theorem of algebra, the polynomial (4) can be factorized into first and second order polynomials in $\cos\omega$, as follows (where $n + 2m = N$, the filter order):

$$H(\omega) = k \cdot \prod_{i=1}^{n}(\cos\omega + a_i) \cdot \prod_{j=1}^{m}(\cos^2\omega + a_{1j}\cos\omega + a_{2j}) \qquad (5)$$

3. CIRCULARLY-SYMMETRIC 2D FIR FILTERS

We describe in this section an efficient design technique for 2D circularly-symmetric filters, based on the previous 1D filters, considered as prototypes.

3.1. Circular Filters Derived from 1D Prototypes

Given a 1D prototype with transfer function $H_P(\omega)$, the 2D circular filter function $H_C(\omega_1, \omega_2)$ results applying the mapping $\omega \to \sqrt{\omega_1^2 + \omega_2^2}$:

$$H_C(\omega_1, \omega_2) = H_P\left(\sqrt{\omega_1^2 + \omega_2^2}\right) \qquad (6)$$

The currently-used approximation of the 2D circular cosine function $\cos\sqrt{\omega_1^2 + \omega_2^2}$ is given by the 3×3 array:

$$\mathbf{C} = \begin{bmatrix} 0.125 & 0.25 & 0.125 \\ 0.25 & -0.5 & 0.25 \\ 0.125 & 0.25 & 0.125 \end{bmatrix} \qquad (7)$$

such that we have the following approximation, a particular case of the McClellan transform [5], [6]:

$$\cos\sqrt{\omega_1^2 + \omega_2^2} \cong C(\omega_1, \omega_2) = -0.5 + 0.5(\cos\omega_1 + \cos\omega_2) + 0.5\cos\omega_1 \cdot \cos\omega_2 \qquad (8)$$

In order to obtain a filter with circular symmetry from the factorized 1D prototype function, we simply replace in the transfer function expression (5) $\cos\omega$ with the circular cosine function (8). The filter convolution kernel $\mathbf{A}$ results as:

$$\mathbf{A} = k \cdot \mathbf{A}_{11} * \mathbf{A}_{12} * \ldots * \mathbf{A}_{1n} * \mathbf{A}_{21} * \mathbf{A}_{22} * \ldots * \mathbf{A}_{2m} \qquad (9)$$

where $\mathbf{A}_{1i}$ ($i = 1 \ldots n$) are 3×3 arrays and $\mathbf{A}_{2j}$ ($j = 1 \ldots m$) are 5×5 arrays, given by the expressions:

$$\mathbf{A}_{1i} = \mathbf{C} + a_i \cdot \mathbf{A}_{01}\ ,\ \mathbf{A}_{2j} = \mathbf{C} * \mathbf{C} + a_{1j} \cdot \mathbf{C}_0 + a_{2j} \cdot \mathbf{A}_{02} \qquad (10)$$

where $\mathbf{A}_{01}$ is a 3×3 zero array and $\mathbf{A}_{02}$ a 5×5 zero array with the central element equal to one; $\mathbf{C}_0$ is a 5×5 array obtained by bordering $\mathbf{C}$ (3×3) with zeros. The above expressions correspond to the factors in (5).

In the following sections we approach the design of two particular cases of 2D filters with circular symmetry, namely wide-band low-pass and selective band-pass circular filters.

3.2. Wide-Band Low-Pass Circular Filters

Let us consider as prototype a low-pass analog elliptic filter of order $N = 4$, peak-to-peak ripple $R_P = 0.04$ dB in the pass-band, stop-band attenuation $R_S = 40$ dB and pass-band-edge frequency $\Omega_P = \pi/2$. Its transfer function in s is:

$$H_P(s) = \frac{0.1037 \cdot (s^4 + 19.864 \cdot s^2 + 84.041)}{(s^4 + 3.2041 \cdot s^3 + 8.4315 \cdot s^2 + 13.126 \cdot s + 14.082)} \qquad (11)$$

Using MAPLE or another symbolic computation program and following the design steps described, a polynomial approximation of the magnitude $|H_P(j\omega)|$ is obtained through Chebyshev expansion, which in factorized form looks like (where $y = \cos\omega$):

$$\begin{aligned} |H_P(\omega)| &\cong 48.6 \cdot (y + 0.8491)(y + 0.7717)(y - 1.087) \\ &\cdot (y^2 + 1.9934y + 0.994)(y^2 + 1.0797y + 0.318) \\ &\cdot (y^2 - 0.3849y + 0.1766)(y^2 - 1.2882y + 0.5314) \\ &\cdot (y^2 - 1.9338y + 0.9726) \end{aligned} \qquad (12)$$

This zero-phase prototype has the frequency response magnitude displayed in Fig.1 (a) and has a rather small ripple in the pass band. The frequency response $H_C(\omega_1, \omega_2)$ of the 2D circular filter results simply in a factorized form by substituting $y \to C(\omega_1, \omega_2)$ in the expression (12). The frequency response magnitude and the corresponding contour plot (isopotentials) are displayed in Fig.1 (b) and (c).

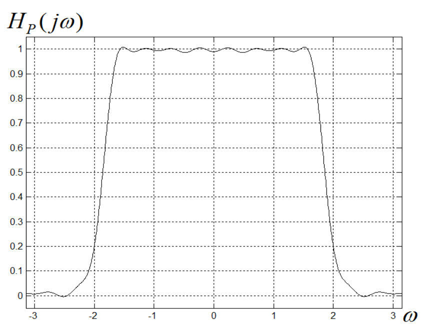
Even if the filter apparently results very complex, i.e. of high order or in other words with a very large convolution kernel (of size 27×27), next we show that using the Singular Value Decomposition (SVD), the designed 2D filter can be reduced in size with a negligible error.

The singular value decomposition of a matrix **M** is written as $\mathbf{M} = \mathbf{U} \times \mathbf{S} \times \mathbf{V}$ where **U** and **V** are unitary matrices and **S** is a diagonal matrix containing the singular values. Thus we can write for the filter convolution kernel **A**:

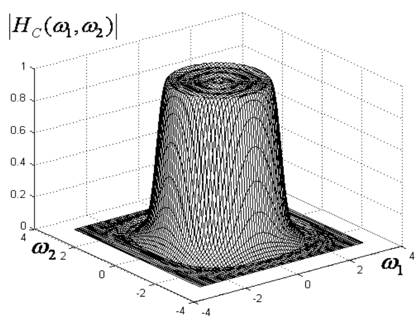
$$\mathbf{A} = \mathbf{U_A} \times \mathbf{S_A} \times \mathbf{V_A} \tag{13}$$

In our case the vector of singular values $\mathbf{S_A}$ of size 1×27 has only 14 non-zero elements:

$$\mathbf{S_{A1}} = [0.50536 \quad 0.086111 \quad 0.032794 \quad 0.013627 \quad 0.00521 \quad 0.002937 \quad 0.001935 \quad 0.001061 \quad 0.000639 \quad 0.000451 \quad 0.000418 \quad 0.0000385 \quad 0.0000196 \quad 0.00000144] \tag{14}$$



(a)



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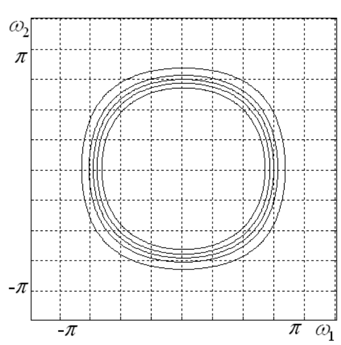
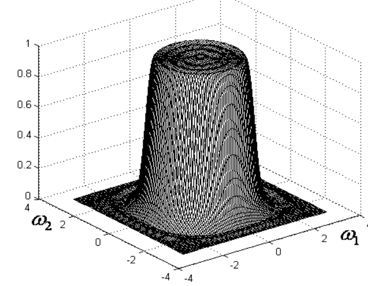
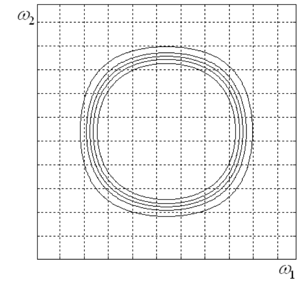


(c)

Fig. 1. (a) Zero-phase low-pass FIR prototype; (b) frequency response magnitude of the 2D circular filter; (c) contour plot;



(a)



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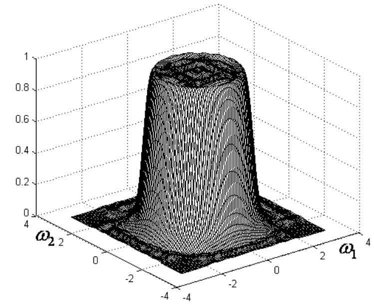

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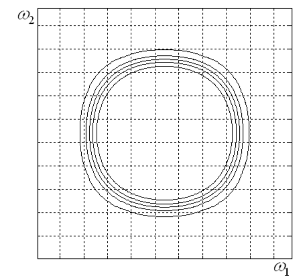


(d)

Fig. 2. (a)-(d) Frequency responses and contour plots for the circular filter preserving only its largest singular values.

Let us denote the vector above as $\mathbf{S_{A1}} = [s_k]$, with $k = 1...14$ in our case. The exact filter matrix **A** can be written as: $\mathbf{A} = \mathbf{U_{A1}} \times \mathbf{S_{A1}} \times \mathbf{V_{A1}}$, where $\mathbf{U_{A1}}$ and $\mathbf{V_{A1}}$ are made up of the first 14 columns of the unitary matrices $\mathbf{U_A}$ and $\mathbf{V_A}$.

If we consider the first largest M values of the vector $\mathbf{S_{A1}} = [s_k]$, the matrix **A** can be approximated as:

$$\mathbf{A} \cong \mathbf{A}_M = \sum_{k=1}^{M} s_k \cdot \mathbf{U_{A}}_k \otimes \mathbf{V}_{\mathbf{A}k}^T \tag{15}$$

Here $\mathbf{A}_M$ is the approximation of matrix **A** taking into account the first M singular values (in our case $M \le 14$), while

$\mathbf{U}_{\mathbf{A}k}$, $\mathbf{V}_{\mathbf{A}k}$ are the k-th columns of matrices $\mathbf{U_{A1}}$ and $\mathbf{V_{A1}}$; the symbol $\otimes$ stands for outer product and superscript T for transposition.

Fig.2 shows the frequency response magnitudes and contour plots for the designed circular filter approximated by taking into account the first largest 8 singular values ((d), (e)) and 5 singular values ((f), (g)).

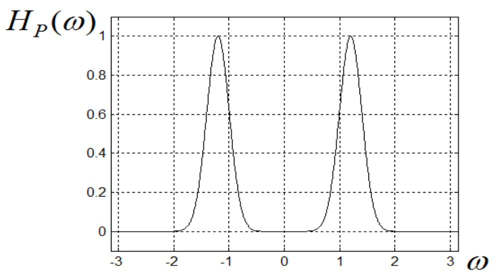
It can be noticed that even retaining only the first 5 singular values, the 2D filter preserves its circular shape without large distortions. In this case the filter template **A** is approximated by $\mathbf{A}_M$ from (15), for $M = 5$. Therefore, the template **A** can be written as a sum of only 5 *separable* matrices according to relation (15). This is an important advantage in filter implementation.

3.3. Selective Band-Pass Circular Filters

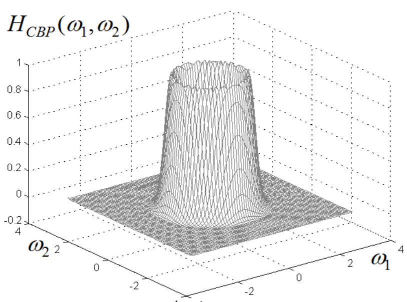
Next we approach the design of another type of zero-phase FIR circular filters, namely very selective band-pass filters. Such a filter may be obtained from an 1D prototype having a Gaussian shape. Let us consider the following prototype function which is the sum of two narrow Gaussians shifted around the frequencies $\pm\omega_0$:

$$H_P(\omega) = \exp\left(-a \cdot (\omega - \omega_0)^2\right) + \exp\left(-a \cdot (\omega + \omega_0)^2\right) \qquad (16)$$

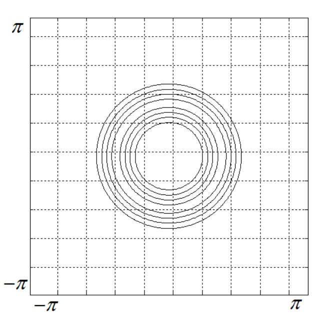
For instance, using the parameter values $a = 12$ and $\omega_0 = 1.2$, we obtain the selective Gaussian-shaped band-pass filter having the real frequency response plotted in Fig.3 (a). Next we follow the same procedure as in the previous section, used for the maximally-flat filters.



(a)

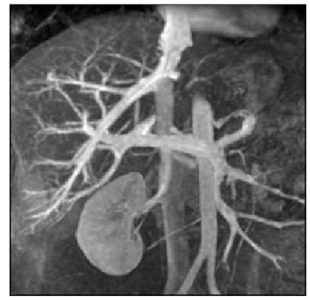


(b)

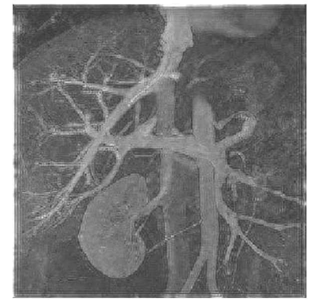


(c)

Fig. 3. (a) Gaussian-shaped selective band-pass 1D prototype; (b) frequency response magnitude of the 2D BP circular filter; (c) contour plot



(a)



(b)

Fig. 4. (a) MRA image; (b) image resulted at the output of the circular band-pass FIR filter

We use again the change of frequency variable (2). As before we must finally obtain a trigonometric series in powers of $\cos\omega$. The result in our case is the following:

$$\begin{aligned} H_P(\omega) \cong\ & 33 \cdot (y + 0.995)(y + 0.95537)(y + 0.87783) \\ & \cdot (y + 0.76577)(y + 0.62419) \cdot (y + 0.45173)(y + 0.354) \\ & \cdot (y - 0.9033)(y - 0.9519)(y - 0.9947)(y - 1.4967) \\ & \cdot (y^2 + 0.3836y + 0.96624) \cdot (y^2 - 1.6875y + 0.7304) \end{aligned} \qquad (17)$$

where $y = \cos(\omega)$. In order to obtain a 2D circularly-symmetric filter, we simply substitute in (17) $y = \cos(\omega)$ with $C(\omega_1, \omega_2)$ with expression (8). The frequency response and contour plot of the 2D BP circular filter are plotted in Fig. 3 (b) and (c).

4. SIMULATION RESULTS

Band-pass filters with given bandwidth and selectivity of the type discussed before can be regarded as concentric components of a circular filter bank, which can be used in image analysis, for instance in texture image decomposition. To show the filtering capabilities of the designed band-pass circular filter, let us consider a medical image shown in Fig.4 (a), namely an angiography image of the liver vein system, obtained through a magnetic resonance angiography (MRA) technique. The image (b) was obtained at the output of the BP circular FIR filter designed in section 3 and shown in Fig. 3. In the output image (b), it can be noticed that both low- and high-frequency components were eliminated. This task can be useful as a pre-processing step in some more complex image processing applications.

5. CONCLUSION

An analytical design method was proposed for low-pass and band-pass zero-phase 2D FIR filters with circular frequency response. Using an efficient polynomial approximation for the low-pass prototype and an appropriate frequency transformation, the desired 2D filter transfer function results directly in a factorized form, which is an advantage in implementation. The novelty of the proposed method lies in the fact that it uses only variable substitutions and approximations and thus avoids the use of the common frequency mappings between the complex planes of the variables s and z, like the Euler approximations or the bilinear transform, which are known to introduce serious distortions of the filter characteristics in the frequency domain. Moreover, the order of the designed filter can be further reduced using the Singular Value Decomposition (SVD), which allows the filter convolution kernel to be reduced in size with negligible error.

Further work on this topic will focus on generalizing this analytical method to design other types of 2D filters, for instance directional filters, fan-shaped, diamond-shaped filters etc.

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