A MULTIDIMENSIONAL APPROACH TO WAVE DIGITAL FILTERS WITH MULTIPLE NONLINEARITIES

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ABSTRACT

The implementation of nonlinear elements in Wave Digital Filters (WDFs) is usually restricted to just one nonlinear one-port per structure. Existing approaches that aim to circumvent this restriction have in common that they neglect the notion of modularity and thus the reusability of the original Wave Digital concept. In this paper, a new modular approach to implement an arbitrary number of nonlinearities based on Multidimensional Wave Digital Filters (MDWDFs) is presented. For this, the contractivity property of WDFs is shown. On that basis, the new approach is studied with respect to possible side-effects and an appropriate modification is proposed that counteracts these effects and significantly improves the convergence behaviour.

Index Terms— Wave Digital Filters, Multidimensional, Contractivity, Multiple Nonlinearities, Analog Modeling

1. INTRODUCTION

The theory of Wave Digital Filters [1] provides an elegant approach to the real-time capable virtual analog modeling of electrical circuits in that it is a modular concept that translates circuit elements and their interconnection topology to corresponding basic building blocks that can be recombined into a readily computable digital structure. Furthermore, with this approach, paramount properties of the original prototype circuit as passivity (thus stability), robustness and modularity are preserved in the Wave Digital (WD) structure. Here, two concepts are important, shown for the multidimensional generalization:

First, the discretization of the complex frequency variables ψ_ν, $\nu = 1, ..., m$ is realized by means of the *bilinear transform*

$$\psi_\nu = \frac{2}{T_\nu} \frac{z_\nu - 1}{z_\nu + 1} \ , \tag{1}$$

where T_ν represents the unit time step in the ν-th variable t_ν. Second, for an arbitrary electrical port with port resistance $R > 0$, current I and voltage U, so-called *wave variables*

$$\begin{aligned} A &= U + RI \\ B &= U - RI \end{aligned} \ , \tag{2}$$

where A and B denote incident and reflected waves, respectively, are introduced. Note that all variables in capital letters indicate steady-state quantities and correspond to instantaneous quantities denoted in lower case.

In conjunction with the Kirchhoff laws of conservation, equations (1) and (2) allow for a multitude of WD elements to be derived. This includes lossless blocks that take care of the interconnection (e.g. parallel or serial connections) of WD elements, so-called *adaptors*, which basically are implementations of the local scattering matrices. For a comprehensive review, the reader is referred to [1].

Unfortunately, there is a major computational constraint: In general, just a single one-port with a direct reflection, e.g. a static nonlinearity like a diode, can be connected to a WD structure without causing implicit relations, i.e. *delay-free loops*. This limits the applicability of the classic WD approach for a lot of circuit simulations, particularly nonlinear ones. For further reading, [2] gives an in-depth overview on the topic. There are prior specialized approaches to the realization of WDFs with multiple static nonlinearities, e.g. [3], [4], but these approaches have in common that they require the network to be decomposed and reformulated, thus omitting the modularity property of the original WD approach.

In this paper, a modular approach to implement an arbitrary number of WD nonlinearities is presented which is derived from MDWDF theory and utilizes a fixed point iteration scheme. For this purpose, first the contractivity properties of WDFs are studied. Then, the new approach is analysed and a modified version with much improved convergence characteristics is presented. Finally, a concrete example is given and analysed.

2. CONTRACTIVITY OF WAVE DIGITAL FILTERS

The contractivity property of general Wave Digital Filters has been presented in [5] in a different context first and is repeated here for better readability:

Given a metric space (M, d) with metric d, a mapping $\varphi : M \to M$ is called *contraction*, if there exists a $\lambda \in [0, 1[$

with the property that for all $x, y \in M$ the *Lipschitz condition*

$$d(\varphi(x), \varphi(y)) \leq \lambda \cdot d(x, y) \tag{3}$$

holds. It can be shown that such self-mappings converge to a unique fixed point $x^* \in M$ under iteration [6], which solves the implicit relation $\varphi(x^*) = x^*$. Similarly, if equation (3) just holds for $\lambda \in [0, 1]$, φ is called a *nonexpansive map*, which is weaker in that it does not allow conclusions about the existence of fixed points. For a deeper insight into the topic, the reader is referred to textbooks like [6].

To investigate the contractivity properties of nonlinear WDFs it is feasible to first analyse a strictly linear foundation WDF: Let A_i and B_i, $i = 1, \ldots, n$, be the incident and reflected steady-state waves, respectively, of an n-port MDWDF network with corresponding finite port resistances $R_i > 0$. For practical reasons this circuit is assumed to be passive but not lossless. Let $\boldsymbol{A} = (A_1 \cdots A_n)^T$, $\boldsymbol{B} = (B_1 \cdots B_n)^T$, $\boldsymbol{G} = diag\left(\sqrt{G_1}, \ldots, \sqrt{G_n}\right), G_i = 1/R_i$ and $\varphi(\boldsymbol{A}) = \boldsymbol{B}$ be the linear map that associates incident and reflected waves.

Based on these assumptions, for the L^2-norm and the Mahalanobis metric $d_{\boldsymbol{G}}(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{(\boldsymbol{x} - \boldsymbol{y})^T \boldsymbol{G}^T \boldsymbol{G} (\boldsymbol{x} - \boldsymbol{y})}$ with positive definite $\boldsymbol{G}^T \boldsymbol{G}$, the steady-state pseudopower P absorbed by that n-port according to [7], [1] can be written as

$$P = \sum_{i=1}^{n} \left(|A_i|^2 - |B_i|^2\right) G_i \overset{!}{>} 0 \tag{4}$$

$$\Leftrightarrow P = \|\boldsymbol{GA}\|_2^2 - \|\boldsymbol{GB}\|_2^2 > 0 \tag{5}$$

$$\Leftrightarrow \|\boldsymbol{GA}\|_2^2 > \|\boldsymbol{GB}\|_2^2 = \|\boldsymbol{G}\varphi(\boldsymbol{A})\|_2^2 \ , \tag{6}$$

and with $\boldsymbol{A} := \boldsymbol{A'} - \boldsymbol{A''}$ and the linearity of φ we have

$$\Leftrightarrow \|\boldsymbol{G}(\boldsymbol{A'} - \boldsymbol{A''})\|_2^2 > \|\boldsymbol{G}(\varphi(\boldsymbol{A'}) - \varphi(\boldsymbol{A''}))\|_2^2 \tag{7}$$

$$\Leftrightarrow \quad d_{\boldsymbol{G}}(\boldsymbol{A'}, \boldsymbol{A''})^2 > d_{\boldsymbol{G}}(\varphi(\boldsymbol{A'}), \varphi(\boldsymbol{A''}))^2 \tag{8}$$

$$\Leftrightarrow \quad d_{\boldsymbol{G}}(\boldsymbol{A'}, \boldsymbol{A''}) > d_{\boldsymbol{G}}(\varphi(\boldsymbol{A'}), \varphi(\boldsymbol{A''})) \tag{9}$$

$$\Leftrightarrow \quad \begin{aligned} &\exists \lambda \in [0, 1[: \\ &\lambda \cdot d_{\boldsymbol{G}}(\boldsymbol{A'}, \boldsymbol{A''}) \geq d_{\boldsymbol{G}}(\varphi(\boldsymbol{A'}), \varphi(\boldsymbol{A''})) \ . \end{aligned} \tag{10}$$

Clearly, an arbitrary lossy, linear WDF is contractive and thus, for constant input, converges towards a unique fixed point under iteration. Note that in order to include lossless circuits there exists a similar relationship with substitution of "$\geq$" in (4), which corresponds to a nonexpansive map in (10). Furthermore, in the analogous description of pseudopower for instantaneous waves a_i and b_i and this time a nonlinear φ, equations (4) and (10) are generally just related by "$\Leftarrow$", so passivity is a result of contractivity in this nonlinear case, but not the other way around.

From here, similar to the interconnection considerations with respect to passivity in [7], this contractive foundation n-port can be extended with further contractive or nonexpansive (analogous to passive and lossless, respectively) structures without losing the contractivity property[1], as a compo-

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[1] with the exception of degenerated cases where a lossless structure is connected formally but the resulting structure is equivalent to at least two mutually independent systems

sition of contractions and nonexpansive maps remains contractive. So the composition of contractive and nonexpansive WD elements always yields a contractive WD structure as long as there is at least one lossy and thus contractive element present, forcing the Lipschitz constant of the system to drop below 1. This includes general multi-port nonlinear contractions as well, that, in particular, may consist of multiple nonlinear one-ports. Note though that a direct proof for general nonlinear WDFs is not possible by means of the instantaneous and steady-state pseudopowers p and P as in [7], [1], respectively, since there is neither a steady-state description for memoryless nonlinear elements nor an appropriate instantaneous representation for frequency-dependent elements.

3. NONLINEAR WAVE DIGITAL FILTERS

As mentioned in [2, 4], there are numerous approaches to adopt nonlinear circuit elements to Wave Digital Filters. With the findings of section 2, one approach certainly becomes more interesting: As a WDF typically has just one reflection-free port that can explicitly handle the direct reflection of a WD nonlinearity, just take the implicit correspondence that occurs by connecting a nonlinearity to a reflecting port and break the computability issue by inserting a delay element (similar to [8]). As has been shown, if the circuit is contractive and the nonlinearity is at least nonexpansive, the resulting structure does converge under iteration and, in absence of reactive components, actually approaches the correct limit value for constant input. But as the following example shows, negative side-effects are likely to occur if further delays (e.g. reactances) are present or if the input varies over time: An ideal diode can be expressed losslessly (and nonexpansively) as $b = -|a|$ in WD terms [9]. Appending a delay denoted by its unit time step T_ν before or after this nonlinear function actually resembles the WD counterparts of a capacitance for $a < 0$ and an inductance for $a > 0$ with respect to a complex frequency ψ_ν. Depending on the circuit, this pseudo-reactive behaviour obviously may introduce significant amounts of error for time-varying input.

3.1. Multidimensional Approach

As has been presented in [5, 10], in a multidimensional sense, the direction of this artificial delay is not bound to a certain dimension. As it serves the sole purpose to resolve a non-computable implicit relation but otherwise may just introduce undesired pseudo-reactive artifacts, it is of particular interest to keep the amount of this error at a low level. To achieve that, for an $(m-1)$-dimensional WDF, the introduction of an additional, artificial m-th dimension is considered. Here, all inputs of the circuit are kept constant with respect to this new dimension and delay elements with time step T_m are introduced at all (again nonexpansive) nonlinearities that are connected to reflecting ports. The unit delay in t_m-direction is di-

mensionless, thus $T_m = 1$. The resulting fixed point iteration along the t_m direction solves the underlying implicit equations and this solution may, after falling below a given error threshold, be read out on the hyperplane $t_m = D \cdot T_m > 0$.

Unfortunately, this approach has drawbacks: Convergence may be very slow and the computational complexity worsens D-fold. Additionally, as shown later in section 4, the further the simulation advances along the main axes, the longer it takes the system to converge with respect to t_m. This is due to the system's dynamic that is applied even to intermediate steps that still have large amounts of error and therefore distributes this undesired information.

3.2. Improved Multidimensional Approach

To improve upon the convergence characteristics of this straight MDWDF approach, a nonlinear modification of the delay elements used is suggested. The basic notion behind this is to stop the aforementioned system's dynamic on intermediate values to make it independent with respect to the simulation's duration and to fasten up convergence by utilizing its contractivity property. While being extendable to multidimensional systems in principle, this nonlinear approach depends on the processing sequence here and thus is not available in closed-form. Therefore, it is presented for a one-dimensional prototype system that is extended by an artificial dimension $m = 2$ to solve the original computability problem.

First, assume a constant simulation length of D steps with respect to the artificial t_2-direction, whereas the time t_1 remains unbounded. To cut the unwanted dynamic, the values of the delays in t_1-direction are simply held constant along the t_2-axis according to

$$b_{T_1}(k_1, k_2) = a_{T_1}(k_1 - 1, D - 1),$$
$$k_2 = 0, \ldots, D - 1, \tag{11}$$

where a_{T_1} and b_{T_1} denote the delay's in- and output, respectively, which corresponds to a discrete sample and hold element that is refreshed every D steps. Note though that its values are sampled at $k_2 = D - 1$, where the system's state, per construction, already should have approached an equilibrium state. This way, a clean fixed point iteration scheme in t_2-direction is made possible: Here, by re-iterating the WD structure, a fixed point is approached and fed back via

$$b_{T_2}(k_1, k_2) = \begin{cases} a_{T_2}(k_1, k_2 - 1) & , k_2 = 1, \ldots, D - 1 \\ a_{T_2}(k_1 - 1, D - 1) & , k_2 = 0 \end{cases}$$
$$\tag{12}$$

as a starting value to the next t_1-sample's fixed point iteration. Due to the nature of contractive mappings, a reasonably close starting value may fasten up convergence a lot. So even if D is chosen too small and convergence has a significant remaining error, the starting value for the next sample in general is better than some constant boundary value. With this approach, convergence can be balanced between a longer iteration length D

or more error along the time axis t_1. Obviously, for constant input and perfect convergence, the starting value is already the solution of the fixed point scheme at any t_1-step. Similar, for low-frequency inputs it is already close, leading to a fast convergence, which is further helped by anti-aliasing techniques like upsampling. Note that a sample and hold element in its hold state, replacing the delay element of e.g. a capacitor, corresponds to a WD resistive source, where the input to the element is omitted and a constant source value is fed back. As the system's contractivity is a global property, it is invariant with respect to the addition of a constant and so the Lipschitz condition (3) holds here as well.

Clearly, the proposed modifications are not passive in the conventional sense, but are at least lossless for constant input. And since it has been shown that the circuit settles towards a constant state of equilibrium along the iteration axis t_2, instabilities are unlikely to occur in that case.

4. EXAMPLE CIRCUIT
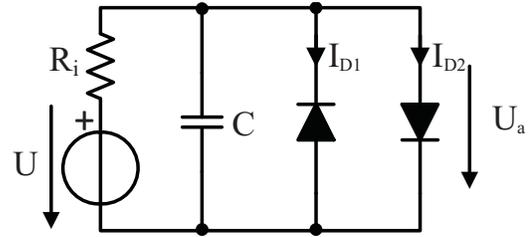
Fig. 1. Prototype circuit

To give a simple example to the presented approach, a small diode clipper circuit as presented in [11] is analysed. As depicted in Fig. 1, it consists of the parallel connection of a resistive voltage source with voltage U and internal resistance $R_i = 2.2\,k\Omega$, a capacitor $C = 0.47\,\mu F$ and a pair of antiparallel diodes D_1 and D_2 that are, apart from their orientation, identical. The diodes are to be implemented according to the Shockley diode model

$$I_{D_1}(U_a) = I_s \left(e^{\frac{U_a}{n_e U_t}} - 1 \right)$$
$$I_{D_2}(U_a) = -I_s \left(e^{\frac{-U_a}{n_e U_t}} - 1 \right) \tag{13}$$

with values $I_s = 2.52\,nA$, $U_t = 26\,mV$, and emission coefficient $n_e = 1.752$. This reference circuit is impossible to build with classic Wave Digital Filters without further considerations due to the fact that typically, a WDF may attach to one static nonlinear element only.

4.1. Reference Model

But this prototype circuit has the interesting property that with a trick, a readily computable WDF may be derived. As the two nonlinear elements are directly connected, they can be regarded as one combined nonlinearity in this special case,

| (a) Reference WDF | (b) Lumped multidimensional WDF | (c) Improved multidimensional WDF |
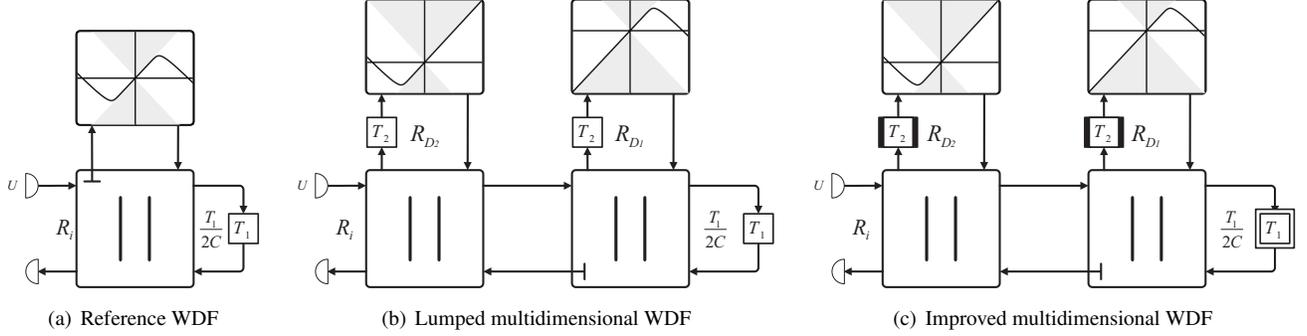
Fig. 2. Wave Digital structures derived from the prototype circuit. (a) Reference implementation, combining the two parallel diodes. Grey areas denote non-passive regions. (b) Multidimensional implementation with artificial delays T_2. (c) Improved multidimensional implementation, where the modifications are denoted by double borders for delay T_1 (now a discrete sample and hold element) and bold sides for T_2 (which now take their start values from the last t_2-iteration of the previous time step), respectively.

thereby neglecting the intended modularity of WDFs. The resulting equation

$$I_D(U_a) = I_{D_1}(U_a) + I_{D_2}(U_a) = 2I_s \cdot sinh\left(\frac{U_a}{n_e U_t}\right) \quad (14)$$

may now be pre-calculated and tabulated by substituting the instantaneous wave definitions (analogous to (2)) and solving numerically with respect to an explicit relation $b = f(a)$. The resulting WD structure with singular nonlinearity is depicted in Fig. 2(a), where the wave representation of Eq. (14) is denoted as well. As it is known to be correct, this implementation with renamed output voltage U_{ref} serves as a reference for the presented approach. Note that for almost all circuits with multiple nonlinearities, a reference like this cannot be built with standard WDF techniques.

4.2. Lumped Circuit Model

Similarly, with the pre-calculation of both nonlinearities with respect to equations (13), a multidimensional model of the prototype circuit may be built without combining the diodes by inserting delays T_2, shown in Fig. 2(b). For the inspected range the resulting nonlinearities are clearly nonexpansive. Since there are no reflection-free adaptor ports needed to connect them, the values of the port resistances for D_1 and D_2, R_{D_1} and R_{D_2}, respectively, become free parameters. It is advisable to keep the amount of direct reflection at these ports at a low level, thus to roughly impedance-match with the rest of the circuit. Indeed, the results vary to a great extent with the choice of R_{D_1} and R_{D_2}, producing different convergence characteristics. Here, $R_{D_1} = R_{D_2} = 1/(1/C + 1/R_i)$ has been chosen, as well as an iteration length of $D = 800$ and t_1-sample rate $F_s = 40 kHz$. The resulting quadratic error with respect to the reference U_{ref} is depicted in Fig. 3(a) for a step response with amplitude $10V$. Two interesting observations can be made here: First, there is a distinctive peak where the initial error is distributed further along a diagonal while decaying very slowly. This stream actually arrives at the slice $t_2 = D \cdot T_2$ and $t_1 \approx 20ms$, where it even becomes visible

in the output signal (not shown). Second, at the beginning of each new time step, the boundary conditions (zeros here) necessitate about 50 iterations before the system's state drops back to a base error level, even for constant input.

4.3. Improved Lumped Circuit Model

The presented improved approach has been designed to prevent these fundamental and distributed errors from developing, and, as depicted in Fig. 3(b), accomplishes this aim for the implementation of the example circuit shown in Fig. 2(c). By cutting the dynamic on initial values with still large errors, the diagonal spreading of the original multidimensional approach is suppressed and due to the introduction of already approximated starting values, there is no need to repeatedly iterate at every new time step to get back to the base error level. The latter seems to essentially depend on the accuracy and resolution of the tabulated values of the nonlinearities, which is true to both multidimensional approaches and the reference simulation. In fact, the proposed approach shows a much improved convergence behaviour, which becomes apparent by reducing the number of iterations per time step to a mere $D = 4$, where the amount of error already has dropped to a considerably low level. This time shown for a cosine input with amplitude $4.5V$ and frequency $80Hz$, the output voltage U_a is depicted in Fig. 3(c) and virtually not distinguishable from the reference.

5. CONCLUSION

In the present paper, a multidimensional approach to the implementation of an arbitrary number of nonlinearities in Wave Digital structures has been presented. In comparison to prior methods, this approach has the advantage of maintaining the modularity property of the Wave Digital concept and thus preserves the topology of the simulated structure. To achieve an appropriate theoretical fundament, the contractivity properties of Wave Digital Filters have been studied, showing that

(a) Error of linear multidimensional approach.

(b) Error of proposed multidimensional approach.

(c) Output of the proposed approach for cosine input after $D = 4$ iterations.
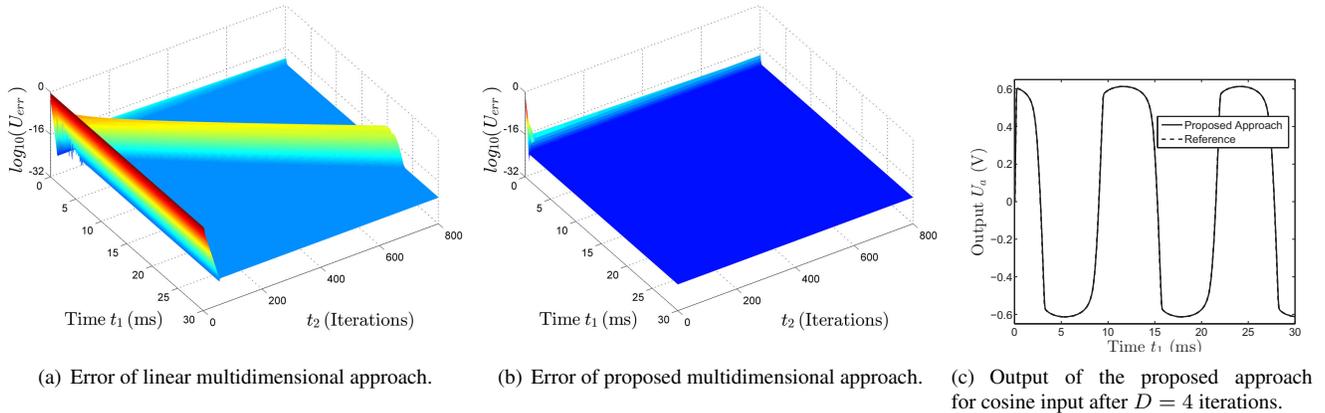
Fig. 3. (a),(b): Quadratic error $U_{err} = (U_a - U_{ref})^2$ of step responses with respect to the reference simulation U_{ref} in a logarithmic scale. Here, input U is an amplified unit step with amplitude $10V$. The error of the unmodified approach in (a) suffers from slowly decaying oscillations across the t_1-t_2-plane as well as from the boundary value problem at the beginning of the t_2-iteration at each new timestep, producing large amounts of error for the first ~ 50 iterations. In (b), the benefit of the proposed improved approach is clearly visible, requiring much smaller numbers of iterations than the chosen $D = 800$ here along both axes to converge. (c) Shows the output voltage U_a of the improved WDF for cosine input with amplitude $4.5V$, frequency $80Hz$ and $D = 4$ iteration steps.

every lossy WD structure is contractive and may be extended with further contractive (or just nonexpansive) elements without losing this property. The contractivity approach to WDFs is similar to the concept of (strict) passivity (and losslessness) known from the general Wave Digital principles, but ensures the existence and uniqueness of a global fixed point. On that basis, general construction considerations to utilize this property have been made and possible negative side-effects have been brought up. To counteract the latter, a modified multidimensional approach has been introduced, basically representing a structurally modular and easy to implement fixed point iteration scheme, which allows the immediate inclusion of multiple nonlinear elements. Finally, a simple example has been given, confirming the former findings.

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