AUTOMATIC OPTIMIZATION OF ADAPTIVE NOTCH FILTER'S FREQUENCY TRACKING

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ABSTRACT

Estimation of instantaneous frequency of narrowband complex sinusoids is often performed using lightweight algorithms called adaptive notch filters. However, to reach high performance, these algorithms require careful tuning. The paper proposes a novel self-tuning layer for a recently introduced adaptive notch filtering algorithm. Analysis shows that, under Gaussian random-walk type assumptions, the resulting solution converges in mean to the optimal frequency estimator. A simplified one degree of freedom version of the filter, recommended for practical applications, is also proposed. Finally, a comparison of performance with six other state of the art schemes is performed. It confirms the improved tracking accuracy of the proposed scheme.

***Index Terms*—** adaptive notch filtering, adaptive signal processing, frequency tracking

1. INTRODUCTION

Consider the problem of estimating an unknown slowly time-varying frequency $\omega(t)$ of a narrowband complex sinusoid (cisoid) using noisy measurements

$$y(t) = s(t) + v(t) \;, \tag{1}$$

where $t = 0,\ 1, \ldots$ denotes discrete time, $v(t)$ is a wideband measurement noise,

$$s(t) = a(t) e^{j \sum_{\tau=1}^{t} \omega(\tau)} \tag{2}$$

is a nonstationaty complex sinusoid with instantaneous frequency $\omega(t)$ and

$$a(t) = m(t) e^{j\phi_0},$$

where $m(t)$ is a real valued slowly time varying amplitude and ϕ_0 denotes initial phase.

Instantaneous frequency of a nonstationary signal may be estimated using a range of tools, e.g. short time Fourier transform (STFT), superresolution methods such as MUSIC or ESPRIT [1], extended Kalman filters [2] adaptive notch filters [3–6] and many others [7]. Regardless of one's particular choice, he or she is immediately faced with the problem of adjusting various algorithm's parameters (such as local analysis window length, adaptation gains, etc.) so as to achieve the optimal, or at least acceptable, frequency tracking accuracy. Since repeated tuning may be necessary (e.g. when the signal to noise ratio changes), it is desirable to extend the estimation tool with means capable of automatic adjustment of its parameters.

The paper develops such a solution for a recently proposed adaptive notch filter. Theoretical analysis shows that, under Gaussian random-walk type assumptions, the proposed scheme locally converges in mean to the optimal values of adaptation gains.

2. SELF TUNING ADAPTIVE NOTCH FILTER

2.1. ANF algorithm and its properties

We will start with the following ANF algorithm, introduced in [8][1]

$$\begin{aligned}
\hat{f}(t) &= e^{j[\hat{\omega}(t-1)+\hat{\alpha}(t-1)]} \hat{f}(t-1) \\
\varepsilon(t) &= y(t) - \hat{a}(t-1)\hat{f}(t) \\
\hat{a}(t) &= \hat{a}(t-1) + \theta_3 \hat{f}^*(t)\varepsilon(t) \\
\hat{\alpha}(t) &= \hat{\alpha}(t-1) + \theta_1 \delta(t) \\
\hat{\omega}(t) &= \hat{\omega}(t-1) + \hat{\alpha}(t-1) + \theta_2 \delta(t) \\
\delta(t) &= \mathrm{Im}\left[\frac{\varepsilon(t)}{\hat{a}(t-1)\hat{f}(t)} \right]
\end{aligned} \tag{3}$$

where $\hat{f}(t)$ is a phase term, $\epsilon(t)$ is the prediction error, $*$ denotes complex conjugation, the quantities $\hat{a}(t)$, $\hat{\omega}(t)$ and $\hat{\alpha}(t)$ are the estimates of the signal's complex 'amplitude', instantaneous frequency and instantaneous frequency rate $[\alpha(t-1) = \omega(t) - \omega(t-1)]$, respectively. The parameters $\theta_1 > 0$, $\theta_2 > 0$, $\theta_3 > 0$, $\theta_1 \ll \theta_2 \ll \theta_3$, are small adaptation gains, determining the rates of amplitude adaptation, frequency adaptation and frequency rate adaptation, respectively.

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[1] The original formulation used the symbols γ_α, γ_ω, μ, rather than θ_1, θ_2, and θ_3.

As shown in [8], the algorithm (3) has very good statistical properties. Under the following assumptions:

(A1) Instantaneous frequency drifts according to the 2-nd order random walk (also called integrated random walk)

$$\begin{aligned}\omega(t) &= \omega(t-1) + \alpha(t-1)\\ \alpha(t) &= \alpha(t-1) + w(t)\ ,\end{aligned} \tag{4}$$

where $\{w(t)\}$ forms a stationary zero-mean Gaussian white noise sequence, $w \sim \mathcal{N}(0, \sigma_w^2)$,
(A2) The sequence $\{v(t)\}$, independent of $\{w(t)\}$, is a circular complex Gaussian white noise, $v \sim \mathcal{CN}(0, \sigma_v^2)$,
(A3) The magnitude of the $s(t)$ is constant, $|s(t)| \equiv a_0$,

a proper choice of the gains θ_1, θ_2, θ_3 can turn the algorithm (3) into a statistically efficient frequency tracker.

Unfortunately, in practice it is not easy to tell when the filter (3) reaches the best frequency tracking accuracy. While one may be tempted to evaluate its performance using mean squared prediction error, this usually leads to underperformance – the settings which minizmize prediction errors differ from those which minimize frequency tracking error [8].

In [9] it was shown that frequency tracking accuracy of the filter (3) may be evaluated using mean squared values of the following signal

$$\xi(t) = \frac{1 - 2q^{-1} + q^{-2}}{\theta_2 + (\theta_1 - \theta_2)q^{-1}}\hat{\omega}(t)\ , \tag{5}$$

where q^{-1} denotes the backward shift operator, $q^{-1}u(t) = u(t-1)$. Note that $\xi(t)$ may be obtained without any prior knowledge of the true values of $\omega(t)$. This makes (5) useful for on-line optimization.

2.2. Self tuning mechanism

Taking advantage of (5) we will design an automatic tuning scheme with an aim of minimizing the following measure of fit

$$J(t) = J(t; \boldsymbol{\theta}) = \frac{1}{2}\xi^2(t)\ , \tag{6}$$

where $\boldsymbol{\theta} = [\theta_1\ \theta_2\ \theta_3]^{\mathrm{T}}$ is the vector of adaptation gains.

Our design will employ the stochastic gradient approach

$$\hat{\boldsymbol{\theta}}(t+1) = \hat{\boldsymbol{\theta}}(t) - g\frac{\partial J(t)}{\partial \boldsymbol{\theta}}\ , \tag{7}$$

where $g > 0$ is a small gain,

$$\frac{\partial J(t)}{\partial \boldsymbol{\theta}} = \left[\frac{\partial J(t)}{\partial \theta_1}\ \frac{\partial J(t)}{\partial \theta_2}\ \frac{\partial J(t)}{\partial \theta_3}\right]^{\mathrm{T}} = \xi(t)\boldsymbol{x}(t)$$

and $\boldsymbol{x}(t) = [x_1(t)\ x_2(t)\ x_3(t)]^{\mathrm{T}} = \partial\xi(t)/\partial\boldsymbol{\theta}$.

To arrive at analytical expressions for $x_1(t)$, $x_2(t)$, $x_3(t)$ the following result from [9] can be applied. Let $e(t) = -\mathrm{Im}\ [v(t)s^*(t)/a_0^2]$. Using the approximating linear filter (ALF) method – a linearization approach proposed in [3] for the purpose of analyzing adaptive notch filters – one can show that

$$\hat{\omega}(t) = \frac{\theta_2 + (\theta_1 - \theta_2)q^{-1}}{D(q^{-1}; \boldsymbol{\theta})}u(t)\ , \tag{8}$$

where

$$\begin{aligned}u(t) &= \omega(t) + (1 - q^{-1})e(t)\\ D(q^{-1}; \boldsymbol{\theta}) &= 1 + d_1q^{-1} + d_2q^{-2} + d_3q^{-3}\ ,\end{aligned} \tag{9}$$

$d_1 = \theta_1 + \theta_2 + \theta_3 - 3$, $d_2 = 3 - 2\theta_3 - \theta_2$ and $d_3 = \theta_3 - 1$.

Using (8) one can rewrite (5) in the following form

$$\begin{aligned}\xi(t) = &-d_1\xi(t-1) - d_2\xi(t-2) - d_3\xi(t-3)\\ &+ u(t) - 2u(t-1) + u(t-2)\ .\end{aligned} \tag{10}$$

Differentiating (10) and using basic facts from the linear filtering theory leads to the below recursive equations

$$\begin{aligned}x_1(t) &= -d_1x_1(t-1) - d_2x_1(t-2) - d_3x_1(t-3)\\ &\quad - \xi(t-1)\\ x_2(t) &= x_1(t) - x_1(t-1)\\ x_3(t) &= x_2(t) - x_2(t-1)\ .\end{aligned} \tag{11}$$

Combining all the partial results derived so far, i.e. (3), (7) and (11), one can obtain the sequential self-tuning adaptive frequency tracker summarized in Table 1. In the next section we will show that, under Gaussian assumptions, the proposed extenion makes the adaptation gains converge in mean to their optimal values.

3. CONVERGENCE ANALYSIS

3.1. Associated ODE

Under small values of g, behavior of the proposed algorithm can be studied by analyzing properties of its associated ordinary differential equation (ODE). Let Ω_{s} denote the stability region of the proposed scheme and $\{\xi(t; \boldsymbol{\theta})\}$, $\{\boldsymbol{x}(t; \boldsymbol{\theta})\}$ denote the stationary processes $\xi(t)$, $\boldsymbol{x}(t)$ which settle down for a constant value of the parameter vector $\boldsymbol{\theta} \in \Omega_{\mathrm{s}}$. Denote by $\boldsymbol{\theta}_*$ the stationary point of the proposed self-optimization loop, i.e. the point in Ω_{s} which satisfies

$$\boldsymbol{F}(\boldsymbol{\theta}_*) = \boldsymbol{0}\ , \tag{12}$$

where

$$\boldsymbol{F}(\boldsymbol{\theta}) = \mathrm{E}[\xi(t; \boldsymbol{\theta})\boldsymbol{x}(t; \boldsymbol{\theta})] \tag{13}$$

and $\mathrm{E}[\cdot]$ denotes expected value.

The linearized ODE associated with the proposed algorithm takes the form

$$\dot{\boldsymbol{X}}_s = -g\boldsymbol{f}(\boldsymbol{\theta}_*)\boldsymbol{X}_s\ , \tag{14}$$

where s denotes continuous time, $\boldsymbol{X}_s = \boldsymbol{\theta}(s) - \boldsymbol{\theta}_*$, and

$$\boldsymbol{f}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{F}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\ . \tag{15}$$

Adaptive frequency tracking

$$\hat{f}(t) = e^{j[\hat{\omega}(t-1)+\hat{\alpha}(t-1)]}\hat{f}(t-1)$$
$$\varepsilon(t) = y(t) - \hat{a}(t-1)\hat{f}(t)$$
$$\hat{a}(t) = \hat{a}(t-1) + \hat{\theta}_3(t)\hat{f}^*(t)\varepsilon(t)$$
$$\hat{\alpha}(t) = \hat{\alpha}(t-1) + \hat{\theta}_1(t)\delta(t)$$
$$\hat{\omega}(t) = \hat{\omega}(t-1) + \hat{\alpha}(t-1) + \hat{\theta}_2(t)\delta(t)$$
$$\delta(t) = \mathrm{Im}\left[\frac{\varepsilon(t)}{\hat{a}(t-1)\hat{f}(t)}\right]$$

Self-tuning

$$\begin{aligned}\xi(t) &= -(1-\hat{\theta}_1(t)/\hat{\theta}_2(t))\xi(t-1)\\ &\quad + 1/\hat{\theta}_2(t)[\hat{\omega}(t) - 2\hat{\omega}(t-1) + \hat{\omega}(t-2)]\\ x_1(t) &= -d_1x_1(t-1) - d_2x_1(t-2) - d_3x_1(t-3)\\ &\quad - \xi(t-1)\\ x_2(t) &= x_1(t) - x_1(t-1)\\ x_3(t) &= x_2(t) - x_2(t-1)\\ \boldsymbol{x}(t) &= [x_1(t)\ x_2(t)\ x_3(t)]^{\mathrm{T}}\\ \hat{\boldsymbol{\theta}}(t+1) &= \hat{\boldsymbol{\theta}}(t) - g\xi(t)\boldsymbol{x}(t)\end{aligned}$$

Table 1. Adaptive frequency tracker with self-tuning.

3.2. Stationary point

Our analysis will be performed under the assumptions (A1)-(A3). Although (A1)-(A3) can be criticized as being somewhat unrealistic, they enable one to perform a meaningful examination of the properties of the proposed scheme. Additional simulation experiments confirm that quantitative results of the below analysis remain valid under relaxed assumptions.

Denote by $\boldsymbol{\theta}_\mathrm{o}$ the vector of optimal gains, i.e. such a value of $\boldsymbol{\theta}$ which minimizes the mean-squared frequency tracking errors,

$$\mathrm{E}[|\hat{\Delta}\omega(t)|^2] \to \min .$$

where $\Delta\omega(t) = \omega(t) - \hat{\omega}(t)$.

We will first show that $\boldsymbol{\theta}_\mathrm{o}$ satisfies (12), i.e. it is a stationary point of the proposed self-tuning loop. Using basic results from the theory of stochastic processes one can arrive at

$$\boldsymbol{F}(\boldsymbol{\theta}) = [F_1(\boldsymbol{\theta})\ F_1(\boldsymbol{\theta})\ F_3(\boldsymbol{\theta})]^{\mathrm{T}}$$
$$F_i(\boldsymbol{\theta}) = \int_0^{2\pi} \frac{e^{j\omega}(1-e^{j\omega})^{i-1}}{D(e^{j\omega};\boldsymbol{\theta})} S_{\xi\xi}(e^{-j\omega};\boldsymbol{\theta})d\omega \qquad (16)$$

where $i = 1,\ 2,\ 3$, $D(e^{j\omega};\boldsymbol{\theta}) = D(q;\boldsymbol{\theta})\big|_{q=e^{j\omega}}$ and, $S_{\xi\xi}(e^{-j\omega};\boldsymbol{\theta})$ is the power spectral density of $\{\xi(t;\boldsymbol{\theta})\}$.

To proceed forward we need the following result [9]: when $\boldsymbol{\theta} = \boldsymbol{\theta}_\mathrm{o}$ the sequence $\xi(t;\boldsymbol{\theta}_\mathrm{o})$ has special interpretation. It holds that

$$\xi(t;\boldsymbol{\theta}_\mathrm{o}) = u(t) - \hat{u}(t|t-1) \ , \qquad (17)$$

where $\hat{u}(t|t-1) = \mathrm{E}[u(t)\,|\,\mathcal{U}(t-1)]$ is the optimal (in the mean-squared sense) one-step-ahead prediction of $u(t)$ based on the set of past observations of $u(t)$, $\mathcal{U}(t-1) = \{u(t-1),\ u(t-2),\ \ldots\}$. Due to this property it can be readily concluded that $\xi(t;\boldsymbol{\theta}_\mathrm{o})$ is white noise (see [10]), i.e. that

$$S_{\xi\xi}(e^{-j\omega};\boldsymbol{\theta}_\mathrm{o}) = \frac{\sigma^2}{2\pi} \ , \qquad (18)$$

where σ^2 depends on σ_e^2 and σ_w^2 in a nontrivial way.

Combining (18) with the first integral of (16) one obtains

$$F_1(\boldsymbol{\theta}_\mathrm{o}) = -\frac{\sigma^2}{2\pi}\int_0^{2\pi} \frac{e^{j\omega}}{D(e^{j\omega};\boldsymbol{\theta}_\mathrm{o})} d\omega \ . \qquad (19)$$

The substitution $z = e^{j\omega}$ converts the integral (19) into the following form

$$F_1(\boldsymbol{\theta}_\mathrm{o}) = -\frac{\sigma^2}{2\pi j}\oint_{\mathcal{C}} \frac{1}{D(z;\boldsymbol{\theta}_\mathrm{o})} dz \ , \qquad (20)$$

where $\mathcal{C}$ denotes the unit circle. Since $\boldsymbol{\theta}_\mathrm{o} \in \Omega_s$ it holds that $D(z^{-1};\boldsymbol{\theta}_\mathrm{o}) = D(q^{-1};\boldsymbol{\theta}_\mathrm{o})\big|_{q=z} = 1+d_1z^{-1}+d_2z^{-2}+d_3z^{-3}$ has all roots inside the unit circle. Consequently, $D(z;\boldsymbol{\theta}_\mathrm{o}) = D(q^{-1};\boldsymbol{\theta}_\mathrm{o})\big|_{q^{-1}=z} = 1 + d_1z + d_2z^2 + d_3z^3$ has all its roots outside the unit circle. Therefore, by Cauchy's residue theorem, $F_1(\boldsymbol{\theta}_\mathrm{o}) = 0$. Similar argument can be used to show that $F_2(\boldsymbol{\theta}_\mathrm{o}) = 0$ and $F_3(\boldsymbol{\theta}_\mathrm{o}) = 0$. This confirms that $\boldsymbol{\theta}_\mathrm{o}$ is a stationary point of the proposed self-tuning mechanism.

3.3. Convergence

Conclusions about convergence of the proposed algorithm can by drawn by studying the matrix $\boldsymbol{f}(\boldsymbol{\theta}_o)$.

After quite tedious calculations it can be shown that [11]

$$\boldsymbol{f}(\boldsymbol{\theta}_\mathrm{o}) = \frac{\sigma^2}{2\pi}\int_0^{2\pi} \boldsymbol{i}(e^{-j\omega})\boldsymbol{i}^{\mathrm{H}}(e^{-j\omega})d\omega$$
$$\boldsymbol{i}_n(e^{-j\omega}) = \frac{(1-e^{-j\omega})^{n-1}}{D(e^{-j\omega};\boldsymbol{\theta}_\mathrm{o})}, \quad n = 1,2,3 \ . \qquad (21)$$

Since the matrix under integral (21) is positive semidefinite for $\omega \neq 0$ and varies nonlinearly with ω, one can conclude that $\boldsymbol{f}(\boldsymbol{\theta}_\mathrm{o})$ is positive definite. This means that (14) is stable.

Under some additional conditions, stated in [12], one obtains the following result:

Proposition For any $\epsilon > 0$ and g sufficiently small, there exists a constant $\delta(g,\ \epsilon)$ such that

$$\limsup_{t\to\infty} P(||\hat{\boldsymbol{\theta}}(t) - \boldsymbol{\theta}_\mathrm{o}|| > \epsilon) \le \delta(g,\ \epsilon)$$

where $\delta(g,\ \epsilon)$ tends to zero as g tends to zero.

Adaptive frequency tracking

$$
\begin{aligned}
\hat{f}(t) &= e^{j[\hat{\omega}(t-1)+\hat{\alpha}(t-1)]}\hat{f}(t-1) \\
\varepsilon(t) &= y(t) - \hat{a}(t-1)\hat{f}(t) \\
\hat{a}(t) &= \hat{a}(t-1) + \hat{\theta}(t)\hat{f}^*(t)\varepsilon(t) \\
\hat{\alpha}(t) &= \hat{\alpha}(t-1) + \hat{\theta}^3(t)\delta(t)/8 \\
\hat{\omega}(t) &= \hat{\omega}(t-1) + \hat{\alpha}(t-1) + \hat{\theta}^2(t)\delta(t)/2 \\
\delta(t) &= \mathrm{Im}\left[\frac{\varepsilon(t)}{\hat{a}(t-1)\hat{f}(t)}\right]
\end{aligned}
$$

Self-tuning

$$
\begin{aligned}
\xi(t) &= -(1-\hat{\theta}(t)/4)\xi(t-1) \\
&\quad + 2/\hat{\theta}^2(t)[\hat{\omega}(t) - 2\hat{\omega}(t-1) + \hat{\omega}(t-2)] \\
x(t) &= -d_1x(t-1) - d_2x(t-2) - d_3x(t-3) \\
&\quad - [1+\hat{\theta}(t) + 3/8\hat{\theta}^2(t)]\xi(t-1) \\
&\quad + [2+\hat{\theta}(t)]\xi(t-2) - \xi(t-3) \\
r(t) &= \rho r(t-1) + x^2(t) \\
\hat{\theta}(t+1) &= \hat{\theta}(t+1) - \xi(t)x(t)/r(t)
\end{aligned}
$$

Table 2. One degree of freedom normalized self-tuning adaptive frequency tracker.

3.4. One degree of freedom algorithm

The algorithm proposed in section 2 can be critized for being somewhat bulky, as it employs three adaptation gains θ_1, θ_2, θ_3. Its simplified version, recommended for practical applications, can be obtained by reducing the number of degrees of freedom from 3 to 1, i.e. by replacing the three gains $\hat{\theta}_1(t)$, $\hat{\theta}_2(t)$, $\hat{\theta}_3(t)$ with a single parameter.

Experience with (3) suggest that the gains θ_1 and θ_2 can be chosen as functions of θ_3 (see [8] for details)

$$\theta_1 = \frac{\theta_3^3}{8} \qquad \theta_2 = \frac{\theta_3^2}{2} \; . \tag{22}$$

The resulting 'preoptimized' algorithm, which additionally employs normalization (the parameter $0 < \rho < 1$ is a user-depandent exponential forgetting constant which govern's the algorithms effective memory length; it is recomended that $\rho \in [0.999; 0.9999]$), is summarized in Table 2

4. SIMULATION RESULTS

4.1. Stationary point

Table 3 compares steady-state means of the parameters $\hat{\theta}_1(t)$, $\hat{\theta}_2(t)$, $\hat{\theta}_3(t)$ with the optimal ones for several values of non-stationarity measure $\kappa = a_0^2\sigma_w^2/\sigma_v^2$ – see [8] for explanation

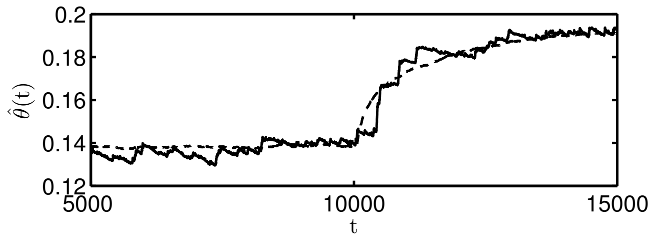

Fig. 1. Typical (solid line) and averaged (dashed line) response of the one degree of freedom scheme to a step change of κ.

of the importance of this quantity. The results were obtained using joint ensemble (50 realizations of $\{w(t)\}$, $\{v(t)\}$, $t \in [0, 100000]$) and time ($t \in [30000, 100000]$) averaging. In the experiment the constant ρ was set to 0.9999. The parameters of the signal were set as follows: $a_0^2 = 10$, random initial phase, $\sigma_v^2 = 1$ and $\sigma_w^2 = \kappa/a_0^2$.

The results confirm that the proposed self-tuning scheme converges in mean to the optimal adaptation gains. Minor discrepancies observed in Table 3 can be attributed to insufficient amount of averaging, which was necessitated by long execution times.

Fig. 1 shows typical and averaged behavior of the simplified scheme in response to a step change of the nonstationarity measure κ from $\kappa_1 = 5 \cdot 10^{-8}$ to $\kappa_2 = 5 \cdot 10^{-7}$ at $t = 10001$ (the constant ρ was set to 0.99925). The proposed scheme reacts to the change properly – by increasing adaptation gain.

4.2. Comparison with existing approaches

Performance of the simplified scheme was compared with other approaches using the following two-mode signal

$$
\omega(t) = \begin{cases} 0.3 & \text{for t} < 3000 \\ 0.2 + 0.1\cos(2\pi(t-3000)/T_\omega) & \text{for t} \geq 3000 \end{cases}
$$
$$a(t) = 3 + \sin(2\pi t/T_a) \; , \tag{23}$$

where $T_\omega = 2000$ and $T_a = 1000$ denote the periods of frequency and amplitude modulations, respectively. The variance of wideband measurement noise was equal to $\sigma_v^2 = 0.01$.

The following algorithms were compared:

- Well tuned Regalia's complex filter [4]. The optimal values of the filter's parameters, i.e. the values which minimized the mean-squared frequency tracking error, were found using exhaustive search.
- Well tuned modified complex plain gradient filter from [5].
- Well tuned arctangent-based complex filter from [6].
- Well tuned ANF (3) with constant gains set according to the rule (22).
- Simplified one-degree of freedom scheme proposed in the paper. The algorithm employed $\rho = 0.999$.

κ	$\mathrm{E}[\hat{\theta}_1(t)]$	$\mathrm{E}[\hat{\theta}_2(t)]$	$\mathrm{E}[\hat{\theta}_3(t)]$	$\theta_{1,\mathrm{o}}$	$\theta_{2,\mathrm{o}}$	$\theta_{3,\mathrm{o}}$
$1 \cdot 10^{-10}$	$1.47 \cdot 10^{-5}$	$1.17 \cdot 10^{-3}$	$4.64 \cdot 10^{-2}$	$1.38 \cdot 10^{-5}$	$1.13 \cdot 10^{-3}$	$4.72 \cdot 10^{-2}$
$1 \cdot 10^{-9}$	$4.43 \cdot 10^{-5}$	$2.45 \cdot 10^{-3}$	$6.82 \cdot 10^{-2}$	$4.32 \cdot 10^{-5}$	$2.41 \cdot 10^{-3}$	$6.85 \cdot 10^{-2}$
$1 \cdot 10^{-8}$	$1.34 \cdot 10^{-4}$	$5.09 \cdot 10^{-3}$	$9.88 \cdot 10^{-2}$	$1.34 \cdot 10^{-4}$	$5.09 \cdot 10^{-3}$	$9.90 \cdot 10^{-2}$
$1 \cdot 10^{-7}$	$4.16 \cdot 10^{-4}$	$1.07 \cdot 10^{-2}$	$1.41 \cdot 10^{-1}$	$4.15 \cdot 10^{-4}$	$1.06 \cdot 10^{-2}$	$1.42 \cdot 10^{-1}$
$1 \cdot 10^{-6}$	$1.25 \cdot 10^{-3}$	$2.20 \cdot 10^{-2}$	$2.00 \cdot 10^{-1}$	$1.26 \cdot 10^{-3}$	$2.19 \cdot 10^{-2}$	$2.01 \cdot 10^{-1}$
$1 \cdot 10^{-5}$	$3.76 \cdot 10^{-3}$	$4.42 \cdot 10^{-2}$	$2.80 \cdot 10^{-1}$	$3.79 \cdot 10^{-3}$	$4.43 \cdot 10^{-2}$	$2.81 \cdot 10^{-1}$

Table 3. Comparison of the steady-state means of the parameters $\hat{\theta}_1(t)$, $\hat{\theta}_2(t)$, $\hat{\theta}_3(t)$ with their optimal values for different values of the nonstationarity measure κ.

Algorithm	$\mathrm{E}[\lvert\Delta\hat{\omega}(t)\rvert^2]$
Regalia's filter	$7.77 \cdot 10^{-7}$
Modified complex plain gradient filter	$7.41 \cdot 10^{-7}$
Arctangent-based complex filter	$5.01 \cdot 10^{-7}$
ANF (3)	$2.69 \cdot 10^{-7}$
Proposed sequential self-tuning estimator	$\mathbf{2.17 \cdot 10^{-7}}$
Sequential self-tuning estimator from [13]	$3.00 \cdot 10^{-6}$

Table 4. Typical mean-squared frequency tracking errors yielded by the proposed one degree of freedom sequential tracker and five other approaches.

- Self-tuning scheme from [13], which adjusts adaptation gains of an ANF so as to minimize mean-squared signal prediction errors yielded by the ANF.

Note that the comparison was fair, if not actually slightly in favor of the constant-parameters schemes which were tuned to yield their best performance. This could be done only because in the experiment the true values of frequency were known. Since such a situation is unlikely to occur in practice, the relative performance of the constant-parameter schemes is somewhat optimistic.

Typical mean-squared frequency tracking errors yielded by the algorithms, evalated for $t \in [1000,\ 5000]$, are summarized in Table 4. The proposed approach clearly outperformed other solutions. The algorithm from [13] performs particularly poorly, but this is not surprising given the fact that it is designed to optimize signal tracking performance.

5. CONCLUSIONS

An automatic tuning mechanism was proposed for an adaptive notch filtering algorithm. Under Gaussian assumptions, the resulting scheme was shown to converge in mean to optimal one. Simulations confirm that the proposed extension of the adaptive filter can improve frequency tracking considerably.

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